Chapter 1: Introduction

Last updated: 8/22/2018

Section	
1.1	Equally Likely Outcomes
1.2	Interpretations
1.3	Distributions
1.4	Conditional Probability and Independence
1.5	Bayes' Rule
1.6	Sequences of Events

Chapter 1 Table of Contents

Chapter 1 Outlines

Section

1.1 Outline: Equally Likely Outcomes

- Set Notations
  $\Omega =$ Outcome Space
  $A =$ Event, which is a subset of Ω
- Equally Likely Outcomes
  Suppose the outcome space, Ω, has equally likely outcomes.
  $$P(A) = \frac{\text{\# of ways that } A \text{ happens}}{\text{\# of possible outcomes}}$$
- Probability to Chance Odds
  Let p be the probability of event A.
  $$P(\text{Event}) = p$$
  Let ω_A to 1 be the *chance odds in favor* of an event A.
  $$\omega_A = \frac{p}{1-p}$$
  Let $\omega_{\bar{A}}$ to 1 be the *chance odds against* an event A.
  $$\omega_{\bar{A}} = \frac{1-p}{p}$$
- Chance Odds to Probability
  If the *chance odds in favor* of an event A are a to b, then
  $$P(A) = \frac{a}{a+b}$$
  If the *chance odds against* an event A are b to a, then
  $$P(\bar{A}) = \frac{b}{a+b} \Rightarrow P(A) = \frac{a}{a+b}$$
- Payoff Odds
  Assume a gambler bets a dollar on an event. The casino's *payoff odds* against an event are a to 1 means that the casino pays the gambler a dollars if the event happens, and the gambler loses a dollar if the event does not happen.
- Fair Odds Rule
  If the payoff odds are equal to the chance odds against an event, then the bet is fair.

1.2 Outline: Interpretations

- Frequency Interpretation
  $P_n(A)$ is the proportion of that event A in n trials.
  $P(A)$ is the probability of A.
  $$P_n(A) \to P(A)$$
- Subjective Interpretation
  For a particular event, repeated trials may not make sense. In this case, the assigned probability may be considered as a *subjective probability* or be called a *probabilistic opinion*.

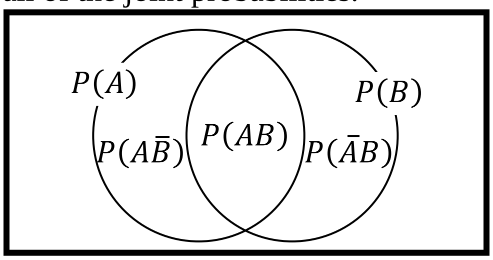
1.3 Outline: Distributions

- Notation

Description	Symbols
A complement Not A	A^C $\bar{A}$
A or B	$A \cup B$
A and B	$A \cap B$

- Venn Diagram

For 2 events, A and B, a Venn diagram displays some of the unconditional probabilities and all of the joint probabilities.



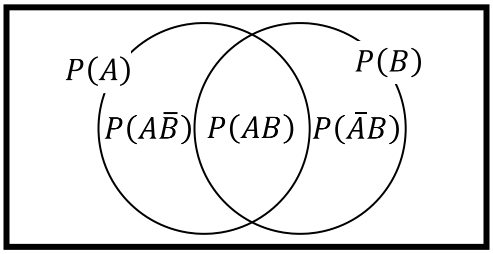
- 2 by 2 Table

For 2 events, A and B, a 2×2 table displays all of the unconditional probabilities (also known as marginal) and all of the joint probabilities.

	B	$\bar{B}$	
A	$P(AB)$	$P(A\bar{B})$	$P(A)$
$\bar{A}$	$P(\bar{A}B)$	$P(\bar{A}\bar{B})$	$P(\bar{A})$
	$P(B)$	$P(\bar{B})$	1

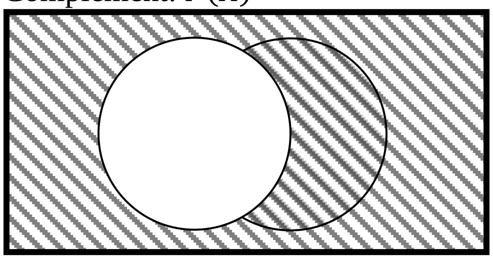
Venn Diagram and 2 by 2 Table

- General



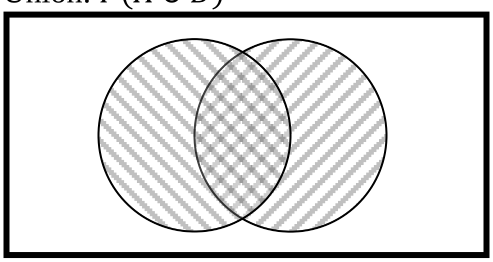
	B	$\bar{B}$	
A	$P(AB)$	$P(A\bar{B})$	$P(A)$
$\bar{A}$	$P(\bar{A}B)$	$P(\bar{A}\bar{B})$	$P(\bar{A})$
	$P(B)$	$P(\bar{B})$	1

- Complement: $P(\bar{A})$



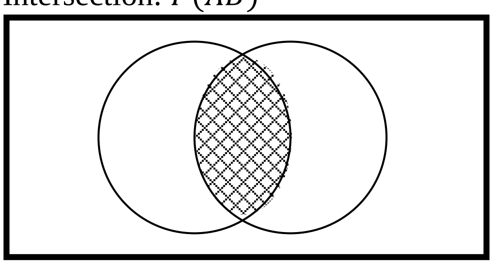
	B	$\bar{B}$	
A	$P(AB)$	$P(A\bar{B})$	$P(A)$
$\bar{A}$	$P(\bar{A}B)$	$P(\bar{A}\bar{B})$	$P(\bar{A})$
	$P(B)$	$P(\bar{B})$	1

- Union: $P(A \cup B)$



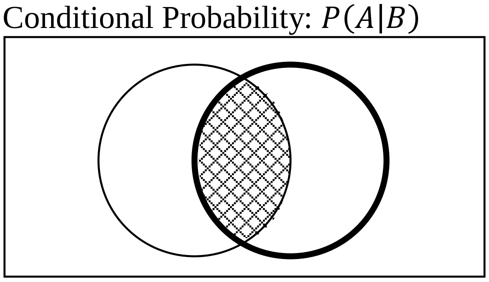
	B	$\bar{B}$	
A	$P(AB)$	$P(A\bar{B})$	$P(A)$
$\bar{A}$	$P(\bar{A}B)$	$P(\bar{A}\bar{B})$	$P(\bar{A})$
	$P(B)$	$P(\bar{B})$	1

- Intersection: $P(AB)$



	B	$\bar{B}$	
A	$P(AB)$	$P(A\bar{B})$	$P(A)$
$\bar{A}$	$P(\bar{A}B)$	$P(\bar{A}\bar{B})$	$P(\bar{A})$
	$P(B)$	$P(\bar{B})$	1

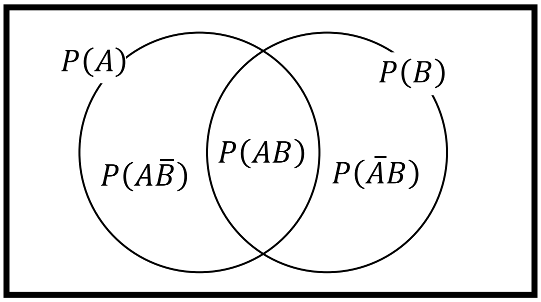
- Conditional Probability: $P(A|B)$



	B	$\bar{B}$	
A	$P(AB)$	$P(A\bar{B})$	$P(A)$
$\bar{A}$	$P(\bar{A}B)$	$P(\bar{A}\bar{B})$	$P(\bar{A})$
	$P(B)$	$P(\bar{B})$	1

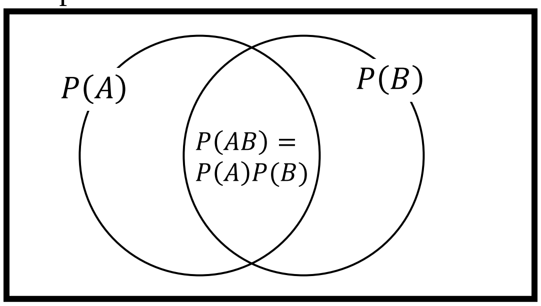
Rules of Probability

- General



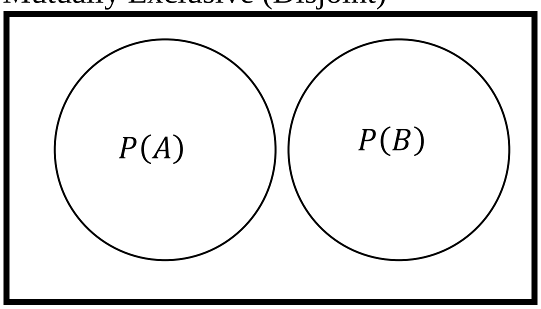
$$P(A \cup B) = P(A) + P(B) - P(AB)$$
$$P(AB) = P(A)B(B|A) = P(B)P(A|B)$$
$$P(A|B) = \frac{P(AB)}{P(B)}, P(B) > 0$$

- Independence



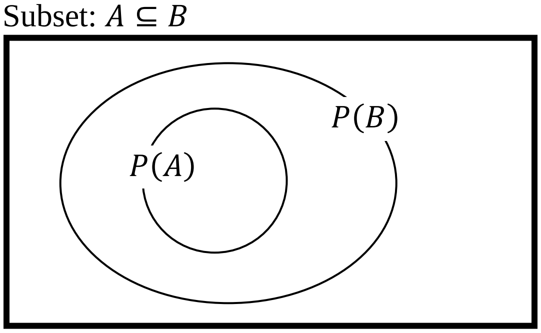
$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$
$$P(AB) = P(A)P(B)$$
$$P(A|B) = P(A)$$

- Mutually Exclusive (Disjoint)



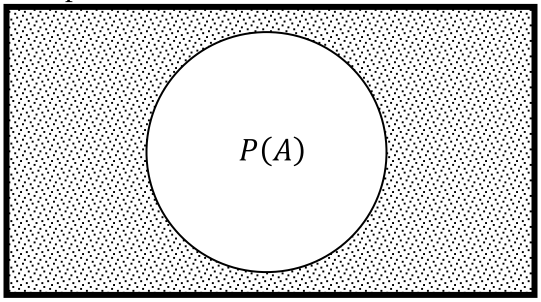
$$P(A \cup B) = P(A) + P(B)$$
$$P(AB) = 0$$
$$P(A|B) = 0$$

- Subset: $A \subseteq B$



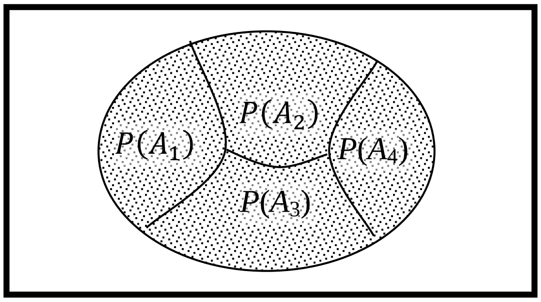
$$P(A \cup B) = P(B)$$
$$P(AB) = P(A)$$
$$P(A|B) = \frac{P(A)}{P(B)}$$
$$P(B|A) = 1$$

- Complement



$P(\bar{A}) = 1 - P(A)$

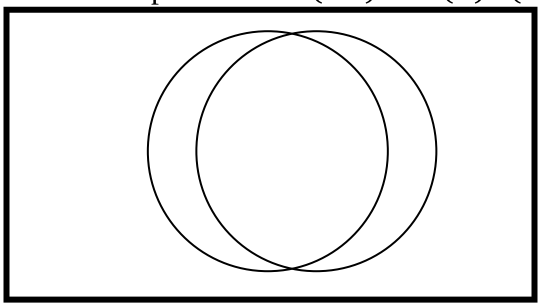
- Partition



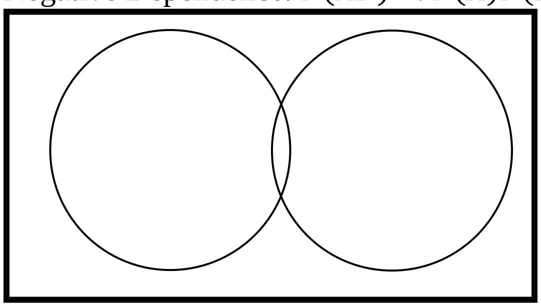
$A_1, A_2, \ldots, A_n$ form a partition of A if $\forall i \neq j\ A_i A_j = \emptyset$ and $\bigcup_{i=1}^{n} A_i = A$.

- Dependence

Positive Dependence: $P(AB) > P(A)P(B)$



Negative Dependence: $P(AB) < P(A)P(B)$



Examples

- Men and getting in an auto accident are *positively* dependent.
- Women and color-blindness are *negatively* dependent.
- Gender and blue eyes are independent.

Union of 3 Events

<u>Inclusion - Exclusion</u>

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

<u>Partition</u>

$$P(A \cup B \cup C) = \begin{cases} P(ABC) + P(AB\bar{C}) + P(A\bar{B}C) + P(A\bar{B}\bar{C}) \\ \qquad +P(\bar{A}BC) + P(\bar{A}B\bar{C}) + P(\bar{A}\bar{B}C) \end{cases}$$

<u>Complement</u>

$P(A \cup B \cup C) = 1 - P(\bar{A}\bar{B}\bar{C})$

General Inclusion-Exclusion Rule

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) - \cdots + (-1)^{n+1} P(A_1A_2 \ldots A_n)$$

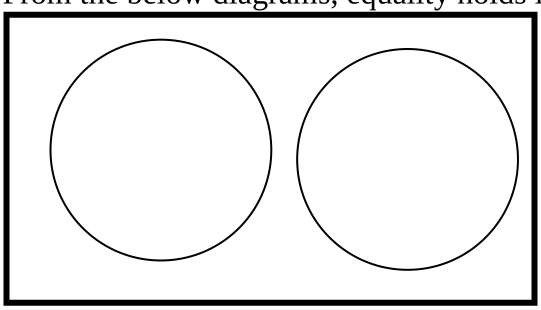
De Morgan's Laws

	Union	Intersection
2	$P(A \cup B) = 1 - P(\bar{A}\bar{B})$	$P(AB) = 1 - P(\bar{A} \cup \bar{B})$
3	$P(A \cup B \cup C) = 1 - P(\bar{A}\bar{B}\bar{C})$	$P(ABC) = 1 - P(\bar{A} \cup \bar{B} \cup \bar{C})$
$\vdots$		
n	$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - P\left(\bigcap_{i=1}^{n} \bar{A}_i\right)$	$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\bigcup_{i=1}^{n} \bar{A}_i\right)$

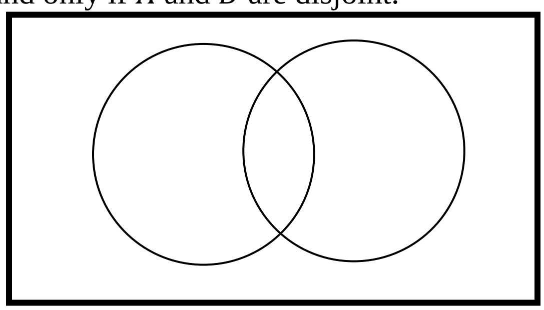
Boole's Inequality

$P(A \cup B) \leq P(A) + P(B)$

From the below diagrams, equality holds if and only if A and B are disjoint.



$P(A \cup B) = P(A) + P(B)$



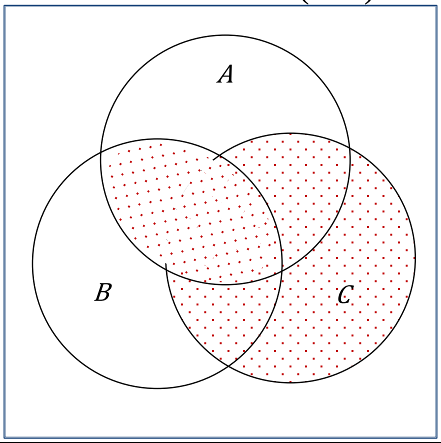
$P(A \cup B) \leq P(A) + P(B)$

Similarly, $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.

In general,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Intersection of Unions (Fact)

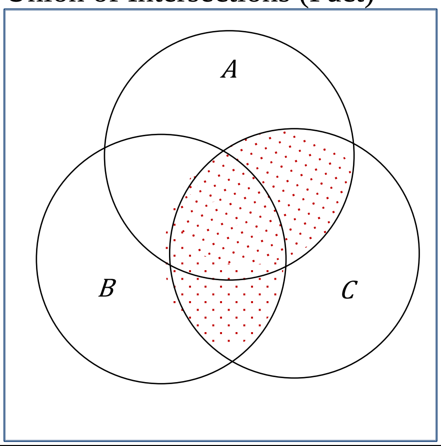


$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

In general,

$$P\left[\left(\bigcap_{i=1}^{n} A_i\right) \cup A_{n+1}\right] = P\left[\bigcap_{i=1}^{n} (A_i \cup A_{n+1})\right]$$

Union of Intersections (Fact)



$(A \cup B) \cap C = AC \cup BC$

In general,

$$P\left[\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right] = P\left(\bigcup_{i=1}^{n} A_i A_{n+1}\right)$$

Bernoulli Distribution

$X \sim Bern(p)$ on $\{0,1\}$

$P(X = x) = p^x (1-p)^{1-x}$

x	$P(X = x)$
0	$1 - p$
1	p

Finite Uniform Distribution from 1 to n

$X \sim Unif(1, \ldots, n)$ on $\{1, \ldots, n\}$

$$P(X = x) = \frac{1}{n}$$

Finite Uniform Distribution from a to b

$X \sim Unif(a, \ldots, b)$ on $\{a, a+1, \ldots, b\}$

$$P(X = x) = \frac{1}{b - a + 1}$$

1.4 Outline: Conditional Probability and Independence

- Multiplication Rule
  $P(AB) = P(A)P(B|A) = P(B)P(A|B)$
  A and B are independent. $\Leftrightarrow P(AB) = P(A)P(B)$
- Conditional Probability
  $P(A|B) = \dfrac{P(AB)}{P(B)}, P(B) > 0$
  A and B are independent. $\Leftrightarrow P(A|B) = P(A)$
- Rule of Average Conditional Probabilities
  Suppose $B_1, B_2, \dots, B_n$ form a partition of B.

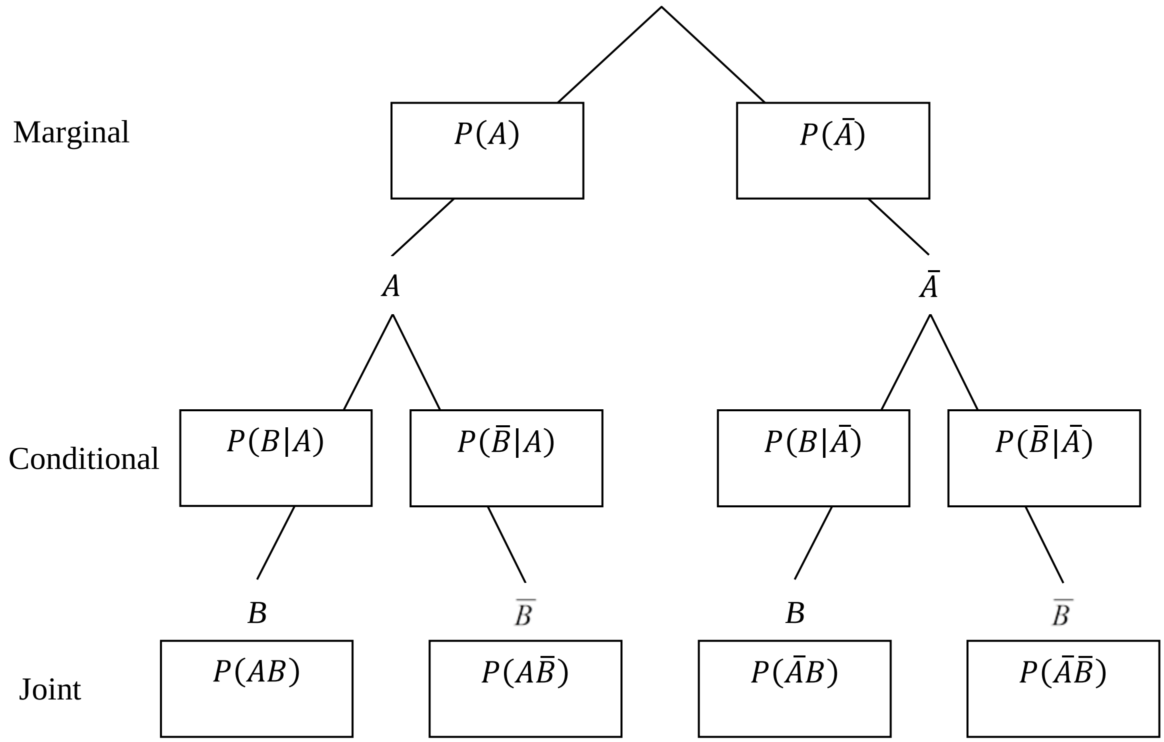
$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

1.5 Outline: Bayes' Rule

- Bayes' Rule

$$P(A_i|B) = \frac{P(A_iB)}{P(B)} = \frac{P(A_iB)}{\sum_{j=1}^{n} P(A_jB)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}, P(B) > 0$$

- Bayes' Tree



Marginal	Conditional	Joint
$P(A)$	$P(B|A)$	$P(AB)$
	$P(\bar{B}|A)$	$P(A\bar{B})$
$P(\bar{A})$	$P(B|\bar{A})$	$P(\bar{A}B)$
	$P(\bar{B}|\bar{A})$	$P(\bar{A}\bar{B})$

1.6 Outline: Sequences of Events

☐ Taylor/Maclaurin Series

Function	Maclaurin Series		Interval
$\dfrac{1}{1-x}$	$=\sum_{i=0}^{\infty} x^i$	$= 1 + x + x^2 + \cdots$	$(-1,1)$
$\log(1+x)$	$=\sum_{i=1}^{\infty} (-1)^{i+1}\dfrac{1}{i}x^i$	$= \frac{1}{1}x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$	$(-1,1]$
e^x	$=\sum_{i=0}^{\infty} \dfrac{1}{i!}x^i$	$= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots$	$(-\infty,\infty)$

☐ Independence of A and B
A and B are independent if and only if $P(AB) = P(A)P(B)$.
The next 3 statements are equivalent.
$P(A\bar{B}) = P(A)P(\bar{B})$
$P(\bar{A}B) = P(\bar{A})P(B)$
$P(\bar{A}\bar{B}) = P(\bar{A})P(\bar{B})$

☐ Multiplication Rule for 3 Events
$P(ABC) = P(A) \cdot P(B|A) \cdot P(C|AB)$

☐ Multiplication Rule for n Events
$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1A_2) \cdot \ldots \cdot P(A_n|A_1A_2 \ldots A_{n-1})$$

☐ Mutual Independence of A, B, and C
A, B, and C are mutually independent if and only if the following:

1. $P(ABC) = P(A)P(B)P(C)$	5. $P(\bar{A}BC) = P(\bar{A})P(B)P(C)$
2. $P(AB\bar{C}) = P(A)P(B)P(\bar{C})$	6. $P(\bar{A}B\bar{C}) = P(\bar{A})P(B)P(\bar{C})$
3. $P(A\bar{B}C) = P(A)P(\bar{B})P(C)$	7. $P(\bar{A}\bar{B}C) = P(\bar{A})P(\bar{B})P(C)$
4. $P(A\bar{B}\bar{C}) = P(A)P(\bar{B})P(\bar{C})$	8. $P(\bar{A}\bar{B}\bar{C}) = P(\bar{A})P(\bar{B})P(\bar{C})$

Verifying all 8 equations is not necessary to prove mutual independence of A, B, and C. However, these 8 statements are not equivalent. Proving a subset of these equations would be sufficient to show mutual independence of A, B, and C.

☐ Sufficient Statements for Mutual Independence of A, B, and C
This set of statements is sufficient to prove A, B, and C are mutually independent.
$P(AB) = P(A)P(B)$
$P(AC) = P(A)P(C)$
$P(BC) = P(B)P(C)$
$P(ABC) = P(A)P(B)P(C)$

- Series

- Geometric Series

  $a =$ first term, $r =$ ratio, $n =$ number of terms

  $\sum_{i=1}^{n} ar^{i-1} = a + ar + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}$

- Infinite Geometric Series

  $a =$ first term, $r =$ ratio where $|r| < 1$

  $\sum_{i=1}^{\infty} ar^{i-1} = a + ar + \cdots = \frac{a}{1 - r}$

- Arithmetic Series

  $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

- Sum of Squares

  $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

- Gambler's Rule

  Suppose in a jar of N tickets, and there is exactly 1 winning ticket. Draw n tickets *with replacement*. Let W be the event of drawing the winning ticket, which means getting the winning ticket at least once in the n trials.

- Find the probability of drawing the winning ticket.

  $p_W = 1 - \left(1 - \frac{1}{N}\right)^n$

  $p_W = 1 - e^{-\frac{1}{N}n}$ for large N

- Find the minimum sample size such that the probability of drawing the winning ticket exceeds p_W.

  $n = \left\lceil \frac{\log(1 - p_W)}{\log\left(1 - \frac{1}{N}\right)} \right\rceil$

  $n = \lceil -\log(1 - p_W) \cdot N \rceil$ for large N

- Birthday Problem

  Draw n tickets with replacement from a jar of N different tickets. Let p be the probability of getting a duplicate.

- Find the probability of getting a duplicate as a function of n.

  $p = 1 - \frac{(N)_n}{N^n}$

  $p = 1 - e^{-\frac{1}{2N}n^2}$ for large N

- Find the sample size required such that the probability of drawing a duplicate is p.

  Plot $f(n) = 1 - \frac{(N)_n}{N^n}$ until $f(n)$ exceeds p.

  $n \geq \sqrt{-2\log(1-p)}\sqrt{N}$ for large N

Chapter 1 Worksheets

1.1 Equally Likely Outcomes	1	2	3	4	5	6	7	8	9	10
	11	12	13	14						
1.2 Interpretations	1	2	3	4	5	6	7	8	9	
1.3 Distributions	1	2	3	4	5	6	7	8		
1.4 Conditional Probability and Independence	1	2	3	4	5	6	7	8		
1.5 Bayes' Rule	1	2	3	4	5	6	7			
1.6 Sequences of Events	1	2	3	4	5	6	7	8	9	10
	11	12	13	14	15					

1.1 Worksheet: Equally Likely Outcomes

1. Flip a fair coin. Find the probability for each of the possible outcomes.

Event	H	T
P(Event)		

2. Roll a fair die.

a) Find the probability for each of the possible outcomes.

Event	1	2	3	4	5	6
P(Event)						

b) Subset the outcomes into the following events.

Prime $\{2, 3, 5\}$ Composite $\{4, 6\}$ Neither $\{1\}$

Find the probability of each event.

3. Roll a fair die twice.

6	1, 6	2, 6	3, 6	4, 6	5, 6	6, 6
5	1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
4	1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
3	1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
2	1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
1	1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
(u_1, u_2)	1	2	3	4	5	6

a) Find the probability for each of the possible sums.

Event	2	3	4	5	6	7	8	9	10	11	12
P(Event)											

b) Roll a fair n-sided die twice. Let k be the possible sums. Find the probability for getting a sum of k. Hint: the range of k is from 2 to $2n$.

4. Roll a fair die 3 times. Find the probability of the following events:

a) 123 in *this* order
b) 321 in this order
c) 1, 2, 3 in *any* order
d) 1, 1, 1 in *any* order
e) 112 in this order
f) 1, 1, 2 in any order

5. Roll a fair die 3 times. Find the probability of the following events:

a) Same roll in the first 2 trials and a unique roll on the last trial
b) Same roll twice and a unique roll (This sequence can occur in any order.)
c) All different numbers
d) All the same numbers

6. Select a ball from a box with 7 red balls and 3 blue balls.
a) Find the chance odds for getting a red ball.
b) Find the chance odds against getting a red ball.

7. Draw 3 cards from a well-shuffled standard deck without replacement. Find:
a) *P*(Ace of Spades, Ace of Hearts, and Ace of Diamonds in this order)
b) *P*(Ace of Spades, Ace of Hearts, and Ace of Diamonds in any order)
c) *P*(an Ace, a Two, and a Three in this order)
d) *P*(an Ace, a Two, and a Three in any order)

8. Draw 3 cards from a well-shuffled standard deck without replacement. Find:
a) *P*(3 of the same rank)
b) *P*(3 different ranks)
c) *P*(2 of the same rank and 1 different rank)

9. Draw 5 cards without replacement from a standard deck. The probability of getting 4 of a kind is $\frac{624}{2{,}598{,}960} = \frac{1}{4{,}165} \approx 0.0002401$. The chance odds against this event are 4164 to 1.
a) I will bet Alex $10 if he gets 4 of a kind. If he doesn't get it, he loses a dime. Is the bet fair?
b) I will bet Courtney $100 if she gets 4 of a kind. If she doesn't get it, she loses a quarter. Is her bet better than Alex? Is the bet fair?

10. Roll a fair die twice. Let $U_1 =$ value of the 1^{st} roll and $U_2 =$ value of the 2^{nd} roll.

6	1, 6	2, 6	3, 6	4, 6	5, 6	6, 6
5	1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
4	1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
3	1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
2	1, 2	2, 2	3, 2	4,2	5, 2	6, 2
1	1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
(u_1, u_2)	1	2	3	4	5	6

Find:
a) *P*(2^{nd} roll is equal to 1^{st} roll) $= P(U_2 = U_1)$
b) *P*(2^{nd} roll is greater than 1^{st} roll) $= P(U_2 > U_1)$
c) *P*(2^{nd} roll is less than 1^{st} roll) $= P(U_2 < U_1)$
d) *P*(1^{st} roll & 2^{nd} roll differ by more than 1) $= P(|U_2 - U_1| > 1)$
e) *P*(2^{nd} roll is *k* greater than the 1^{st} roll) $= P(U_2 - U_1 = k)$
f) *P*(1^{st} roll & 2^{nd} roll are *k* apart) $= P(|U_2 - U_1| = k)$

11. Roll a fair die twice. Let $U_1 =$ value of the 1^{st} roll, $U_2 =$ value of the 2^{nd} roll, $X = \min(U_1, U_2)$ and $Y = \max(U_1, U_2)$. Find:
a) *P*(min is *x* or greater) $= P(X \geq x)$
b) *P*(min is *x*) $= P(X = x)$
c) *P*(max is *y* or smaller) $= P(Y \leq y)$
d) *P*(max is *y*) $= P(Y = y)$

12. Roll a fair die n times. Let $X = \min(U_1, \dots, U_n)$ and $Y = \max(U_1, \dots, U_n)$. Find:
a) $P(X \geq x)$
b) $P(X = x)$
c) $P(Y \leq y)$
d) $P(Y = y)$

13. Roll a fair die n times. Find:
a) P(all sixes)
b) P(no sixes)
c) P(not all sixes)
d) P(at least 1 six)

14. Draw $n = 10$ cards from a well-shuffled standard deck without replacement. Find:
a) P(all hearts)
b) P(no hearts)
c) P(not all hearts)
d) P(at least 1 heart)

1.2 Worksheet: Interpretations

1. State if each probability has a frequency interpretation or a subjective interpretation.
a) P(Four of a kind when drawing 5 cards)
b) P(An earthquake larger than 7.0 Richter scale occurring in the Bay Area this year)
c) P(Thumb tack lands up)
d) P(49ers will win the next Super Bowl)

2. Draw 5 cards without replacement from a standard deck. The probability of getting four of a kind is $\frac{624}{2{,}598{,}960} = \frac{1}{4{,}165} \approx 0.0002401$. A player will bet a quarter hoping to get four of a kind. If he wins, he keeps his quarter and gets \$20 from the casino.
a) Find the chance odds against getting four of a kind.
b) What are the player's payoff odds?
c) Is the bet fair?

3. A roulette wheel has equally 38 numbers which are labeled 1 through 36, 0, and 00. See page 7 in the text for a full description. Suppose a gambler bets on a *quarter play*, which is betting on 4 specific numbers.
a) Find the probability of winning.
b) Find the chance odds in favor of a quarter play.
c) Find the chance odds against a quarter play.
d) What would be the fair payoff odds for a quarter bet?
(In general, you can state the payoff odds in reduced form as some number to 1. So the payoff odds of 5 to 2 can be stated as 2.5 to 1.)
e) Do you expect the actual payoff odds to be less than or greater than the fair payoff odds? Remember that all the lights glittering in Vegas are not built on winners.

4. If the payoff odds for a lottery are 999,999 to 1, the probability of winning should be __________________ than ___________________. Choose *greater* or *less* for the first blank and write a probability for the second blank.

5. Flip a fair coin 10 times.
a) A casino offers you 5 to 1 payoff odds if you get a run of 5 heads or more. In order for this to be a fair bet the probability of getting a run of 5 heads or more should be: ________________
b) In fact this is not a fair bet, because the probability of getting a run of at least 5 heads is $\frac{112}{1024}$. (At this point in this course, you are not expected to know how to calculate this probability.) For every dollar you bet, what is your expected net gain.

6. If the chance odds against an event are 10 to 1, and the payoff odds are 8 to 1, find the expected net gain per dollar bet.

7. A casino offers you 2 to 1 payoff odds to play a game of gin rummy. To keep the game simple, each game results in either a win or a loss where the player bets \$1 each game.
a) If a player has a 0.5 probability of winning a game, find his expected net gain per dollar.
b) After many games, a player finds that his expected net gain per dollar is 0.10. Find the probability that this player wins a game.

8. Suppose there are 3 horses in a race and a bookmaker quotes the following payoff odds against horse i winning. Make an educated guess as to how you can tell that the gambler does not have a sure way of winning. (In Pitman 1.2.3b, you will prove this result.)

Horse i	Payoff odds: r_i to 1	$p_i = 1/(r_i + 1)$.
1	1 to 1	
2	3 to 1	
3	2 to 1	
		$\Sigma = p_1 + p_2 + p_3$

9. Suppose there are 3 horses in a race and a bookmaker quotes the following payoff odds against horse i winning. A gambler has \$3 dollars to bet in any allocation.

Horse i	Payoff odds: r_i to 1	$p_i = 1/(r_i + 1)$.
1	1 to 1	
2	3 to 1	
3	4 to 1	
		$\Sigma = p_1 + p_2 + p_3$

a) What are the possible outcomes if the gambler bets all \$3 dollars on:
   horse 1? horse 2? horse 3?

b) What are the possible outcomes if the gambler bets \$1 dollar on each horse?

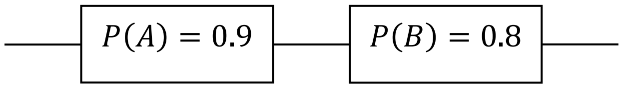
c) How should the gambler allocate his bet such that he always wins the same amount no matter which horse wins?

Horse 1	Horse 2	Horse 3	Win	Net/H1	Net/H2	Net/H3	Net Amt
3	0	0	1				
			2				
			3				
0	3	0	1				
			2				
			3				
0	0	3	1				
			2				
			3				
1	1	1	1				
			2				
			3				
			1				
			2				
			3				

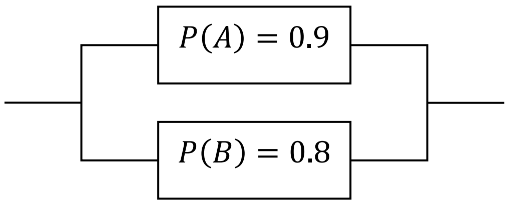
1.3 Worksheet: Distributions

1. Assume the circuits in each system are independent. Use the following notation. Let $P(W)$ be the probability that circuit W is working. Find the probability that the system is working.

a) Circuits in Series



b) Circuits in Parallel



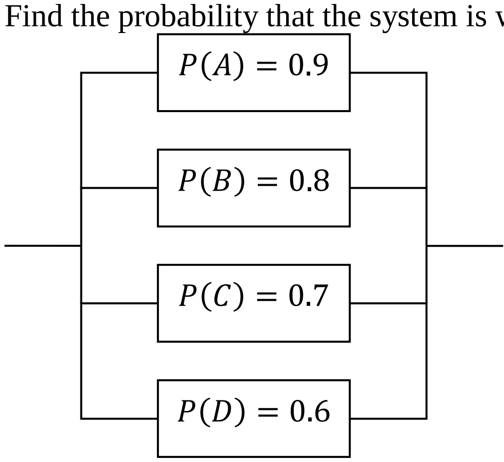
2. Prove the inclusion-exclusion principle for $n = 3$ by each method.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

a) Venn diagram

b) Algebraic proof

3. Find the probability that the system is working assuming mutual independence.



4. Suppose $P(A) = 0.41$, $P(B) = 0.48$, $P(C) = 0.51$, $P(AB) = 0.25$, $P(AC) = 0.15$, $P(BC) = 0.30$, $P(ABC) = 0.10$. Verify $P(A \cup B \cup C)$ by each method.

a) Inclusion - Exclusion

b) Partitions

c) Complement

5. Using a Venn diagram, show the following:

a) $P\big((A \cup B) \cap C\big) = P(AC \cup BC)$

b) $P\big((A \cup B) \cap C\big) \neq P\big(A \cup (B \cap C)\big)$

6. For the given $P(A)$ and $P(B)$, bound $P(AB)$ and $P(A \cup B)$
a) Given $P(A) = 0.32, P(B) = 0.40$, bound $P(AB)$.
b) Given $P(A) = 0.75, P(B) = 0.40$, bound $P(AB)$.
c) Given $P(A) = 0.32, P(B) = 0.40$, bound $P(A \cup B)$.
d) Given $P(A) = 0.75, P(B) = 0.40$, bound $P(A \cup B)$.

7. Write out the terms.
a) $\sum_{i=1}^{4} P(A_i)$
b) $\sum_{1 \le i < j \le 4} P(A_i A_j)$
c) $\sum_{i=1}^{4} P(A_i A_5)$

8. Simplify each expression.
a) $\sum_{i=1}^{n} c$
b) $\sum_{1 \le i < j \le 4} P(A_i A_j) + \sum_{i=1}^{4} P(A_i A_5)$

1.4 Worksheet: Conditional Probability and Independence

1. Suppose $P(A) = 0.40$ and $P(B) = 0.30$. For each intersection probability, indicate if A and B are independent, positively dependent, negatively dependent, or mutually exclusive. Circle the right option: $=$ $>$ $<$.

a) $P(AB) = 0.12$
$\Rightarrow A$ & B are _________________________. $\Leftrightarrow P(A|B)$ $=$ $>$ $<$ $P(A)$

b) $P(AB) = 0.15$
$\Rightarrow A$ & B are _________________________. $\Leftrightarrow P(A|B)$ $=$ $>$ $<$ $P(A)$

c) $P(AB) = 0.10$
$\Rightarrow A$ & B are _________________________. $\Leftrightarrow P(A|B)$ $=$ $>$ $<$ $P(A)$

d) $P(AB) = 0$
$\Rightarrow A$ & B are _________________________. $\Leftrightarrow P(A|B)$ $=$ ________ .

2. If A and B are mutually exclusive, A and B can be thought of as negatively dependent (most of the time). However, there is a special case where A and B are both mutually exclusive and independent. Give an example when this may occur.

3. Deal 2 cards from a standard deck. Circle the right option: =, >, or <.

a) $P(2^{\text{nd}}$ card is a spade) = > < P(1st card is a spade)
b) $P(2^{\text{nd}}$ card is a spade | 1^{st} card is black) = > < $P(2^{\text{nd}}$ card is a spade)
c) $P(2^{\text{nd}}$ card is a spade | 1^{st} card is a clubs) = > < $P(2^{\text{nd}}$ card is a spade)
d) $P(2^{\text{nd}}$ card is a spade | 1^{st} card is a face card) = > < $P(2^{\text{nd}}$ card is a spade)

4. Deal 2 cards from a standard deck. Show $P(2^{\text{nd}}$ card is a spade$) = \frac{1}{4}$ using a tree diagram.

5. Suppose $P(A) = 0.41$, $P(B) = 0.48$, $P(C) = 0.51$, and events A, B, and C are all mutually independent. Find $P(A \cup B \cup C)$.

6. A jar has 3 cards as listed in the table below. Shake the jar and select a card randomly.

Type
rr (both sides are red)
rb (1 side red and 1 side black)
bb (both sides are black)

a) If the card is showing red, find the probability that the other side is black.
b) If the card is showing red, find the probability that the other side is red.

7. A jar has 3 types of coins as listed in the table below. Select a coin from the jar, and flip it twice. Let H_i be the event that the i^{th} toss is a head.

Type	P(Type)
hh (both sides are heads)	0.4
ht (normal coin)	0.2
tt (both sides are tails)	0.4

a) If the coin is showing heads on its 1^{st} toss, find the probability that the other side is also heads.
b) Are H_1 and H_2 independent or dependent?
c) Are H_1 and H_2 independent or dependent conditional on coin type?

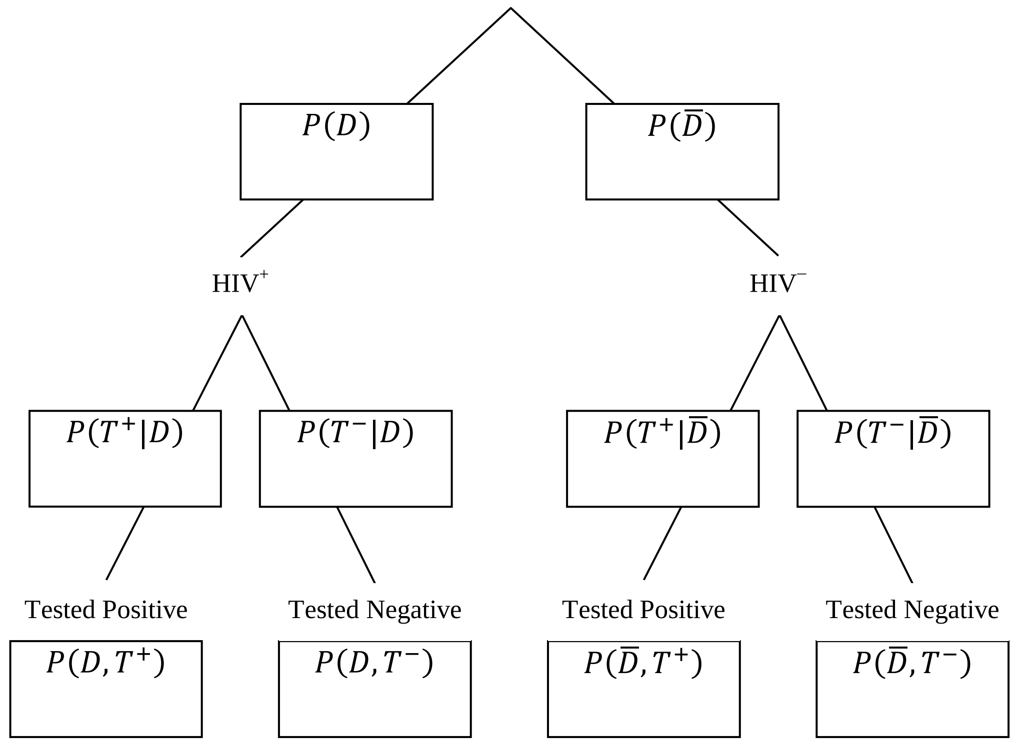
8. A jar has 3 types of coins as listed in the table below. Select a coin from the jar, and flip it twice. Let H_i be the event that the i^{th} toss is a head.

Type	P(Type)
hh (both sides are heads)	0.6
ht (normal coin)	0.3
tt (both sides are tails)	0.1

Find:
a) $P(hh|H_1)$ and $P(ht|H_1)$
b) $P(H_2|H_1)$
c) $P(hh|H_1H_2)$ and $P(ht|H_1H_2)$
d) $P(H_3|H_1H_2)$ and $P(H_3|H_1T_2)$

1.5 Worksheet: Bayes' Rule

1. Suppose 3% are HIV+ in a population. There is a test, which has a 0.98 probability of testing positive if someone is HIV+ and a 0.99 probability of testing negative if someone is HIV-. In epidemiological language, *sensitivity* is the probability of a positive test among the disease individuals, and *specificity* is the probability of a negative test among the non-disease individuals. An individual is randomly selected from this population and is tested for his HIV status.



a) There are 2 ways a misdiagnosis can happen. Find the probability of a misdiagnosis.
b) A person who does not have the disease, but is tested positive is a *false positive*. The *false positive rate* is the probability of not having the disease among those who tested positive. Find the false positive rate.

2. Each morning Jack or Jill walks up the hill alone to fetch a pail of water. Over the course of the year, Jill being the more diligent one walks up the hill 4 times more often than Jack. In 16% of their total attempts to fetch a pail of water, Jack or Jill falls down the hill. Assume they don't get the pail of water whenever they fall down the hill. Since Jill is the more experienced climber, she falls down only 5% of the time.

a) When Jack goes up the hill, find the probability he falls down the hill. Do you expect this number to be larger or smaller than 16% (the probability of falling down the hill)?
b) A pail of water was retrieved. Find the probability that Jack got the water.
c) A pail of water was retrieved. Find the probability that Jill got the water.
d) On this day, someone has fallen down the hill. Find the probability it was Jack.
e) On this day, someone has fallen down the hill. Find the probability it was Jill.

3. Suppose 10% of a population have lung cancer. Those with lung cancer are twice as likely to be smokers as those without lung cancer. Find the probability that a smoker has lung cancer.

4. **Two Cubs Problem**: Oski has 2 cubs. Assume the probability of a girl cub is 0.5 and the gender of the firstborn is independent of the second born. Find the probability that both cubs are girls under each condition.
a) The older one is a girl.
b) Oski tells you that he has at least one girl cub.
c) (Optional) You run into Oski with one of his cubs and you notice that he has a girl. Make whatever assumption you want to answer this question.
d) You run into Oski with one of his cubs and you notice that he has a girl. (Now you know that he has at least 1 girl.) Assume Oski randomly chooses one of his cubs to walk with him.
e) You run into Oski with one of his cubs and you notice that he has a girl. Assume Oski is from a culture where the father always chooses to walk with his girl cub instead of his boy cub if there is a choice between a girl cub and a boy cub.
f) You run into Oski with one of his cubs and you notice that he has a girl. Assume the probability that Oski chooses to walk his girl cub instead of his boy cub is w when he has choice between a girl cub and a boy cub.

5. **Two Cubs Problem Extended**: Oski has 2 cubs. Assume the probability of a girl cub is 0.5 and the gender of the firstborn is independent of the second born. Find the probability that both cubs are girls under each condition.
a) Oski tells you that he has at least one girl cub that was born on Tuesday.
b) Oski tells you that he has at least one girl cub that is born on September 27. Estimate what this answer should be. Should the answer be closer to $\frac{1}{3}$ or $\frac{1}{2}$. As trivia, Oski was born on Tuesday, September 27, 1941.
c) Oski tells you that he has at least one girl cub that is the oldest.
d) Oski tells you that he has at least one girl cub that eats fish. Assume all cubs eat fish. (This is a strange way of phrasing this question. However, it is equivalent to "Oski tells you that he has at least one girl cub.")

6. Laplace's Rule of Succession
In $n + 1$ trials, there are k successes in the first n trials. Find the probability that the last trial is a success under each assumption.
a) The probability that there are $k + 1$ successes in all $n + 1$ trials is 1.
b) The probability that there are $k + 1$ successes in all $n + 1$ trials is 0.
c) The probability that there are $k + 1$ successes in all $n + 1$ trials is 0.5. See the next problem to understand why a probability of 0.5 can be justified.
d) The probability that there are $k + 1$ successes in all $n + 1$ trials is p.

7. Laplace's Rule of Succession
There are $n + 1$ boxes labeled from $0, 1, 2, \ldots, n$. In each box, there are $n + 1$ marbles. Box 0 has 0 black and $n + 1$ white marbles. Box 1 has 1 black and n white marbles. In general, Box i has i black and $(n + 1 - i)$ white marbles. Pick a box at random, and draw n marbles, so there is one left in the box. Given there are k black marbles in the n draws, find the probability that the one in the box is black.

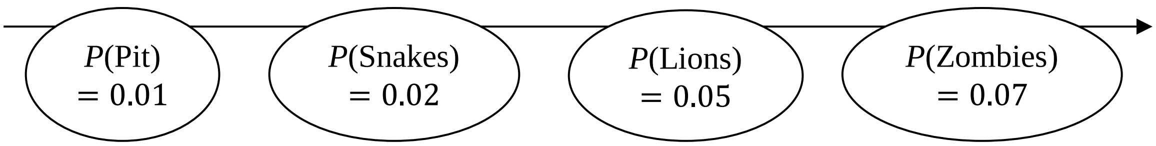
1.6 Worksheet: Sequences of Events

1. Prove the following statements.

a) $P(AB) = P(A)P(B) \Rightarrow P(A\bar{B}) = P(A)P(\bar{B})$

b) $P(ABC) = P(A) \cdot P(B|A) \cdot P(C|AB)$

2. Rick Grimes has to travel a path where he encounters the following perils in sequential order. Listed are the probabilities of death *when encountering* each peril. Assume the events are independent.



a) P(death by falling in the pit)

b) P(death by snake bites along this path)

c) P(death by lions along this path)

d) P(death by zombies along this path)

e) P(survives)

f) P(dies along this path)

g) P(death | escaped the snake bites)

3. **Mutual Independence**

Each person, Alice, Bob, and Carol, rolls a fair die. Let E_{12} be the event that Alice and Bob get the same roll, E_{13} be the event that Alice and Carol get the same roll, and E_{23} be the event that Bob and Carol get the same roll.

a) Are E_{12}, E_{13}, and E_{23} pairwise independent?

b) Are E_{12}, E_{13}, and E_{23} mutually independent?

4. **Gambler's Problem**: $N = 365 \Rightarrow p = \frac{1}{N} = \frac{1}{365}$

a) In a group of n people, find the probability that someone in the group has the same birthday as you assuming no one is born on leap day, February 29.

b) Consider p as small. Use a Taylor series to find the above probability as a function of n.

c) Find the smallest n such that the probability that someone in the group has the same birthday as you exceeds 0.5.

d) Consider p as small. Use a Taylor series to find the n in part c) as a function of N.

5. **Birthday Problem**: $N = 365$

a) In a group of n people, find the probability that at least 2 people have the same birthday.

b) Consider N as large. Use a Taylor series to find the probability in part a) as a function of n.

c) Find the smallest n such that the probability that at least 2 people share the same birthday exceeds 0.5. (Do this problem numerically on a calculator/computer.)

d) Consider N as large. Use a Taylor series to find the n in part c) as a function of N.

6. Evaluate the player's net gain for the each betting option. Which of the following two betting options is better?

a) Option 1: The player bets against any duplicate birthdays out of 60 people. If at least 2 people do not have the same birthday, then the player wins \$$50$. Otherwise she loses \$$0.50$.

b) Option 2: The player bets against any duplicate birthdays out of 40 people. If at least 2 people do not have the same birthday, then the player wins \$$5$. Otherwise she loses \$$0.50$.

7. There are $1{,}000$ tickets in a box numbered from 1 to $1{,}000$. Select n tickets randomly *with replacement*. Ticket #1 is designated as the winning ticket. Let W be the event of getting the winning ticket and D be the event that all the tickets are distinct. Find:

a) Probability of getting the winning ticket

b) Minimum number of tickets drawn such that *P*(winning ticket) exceeds 0.5

c) Probability that there is a repeat

d) Minimum number of tickets drawn such that *P*(repeat) exceeds 0.5

8. Suppose events A, B and C are pairwise independent. The following probabilities are given:

$$P(A) = 0.4,\ P(B) = 0.3,\ P(C) = 0.6,\ P(ABC) = 0.03$$

a) Find the following probabilities:

$$P(AB),\ P(AC),\ P(BC),\ P(AB\bar{C}),\ P(A\bar{B}C),\ P(\bar{A}BC),\ P(\bar{A}\bar{B}\bar{C})$$

b) Are events A, B and C are mutually independent?

9. There are $N = 12$ months in a year. Approximate the probability of being born under each month as equally likely.

a) Find the number of people such that there is at least a 50% probability that 2 or more of them were born on the same month.

b) Ask people until there is a repeat of birthday month. Let $X = \#$ of people. Find $P(X = x)$.

10. Flip a p-coin until you get a head. Let $X = \#$ of trials. Find:

a) $P(X = x)$

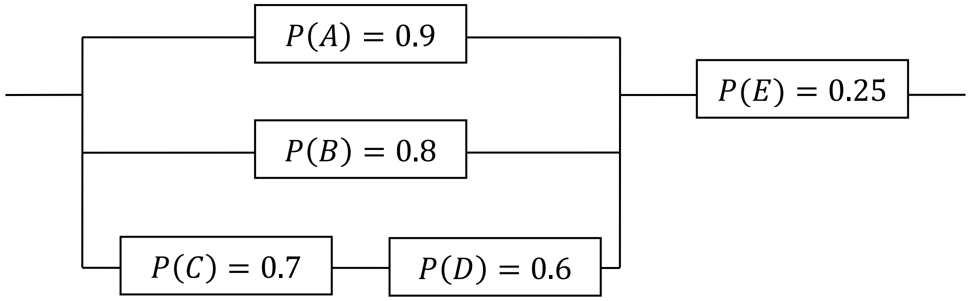
b) $P(X > x)$

11. Roll an n-sided die until there is a duplicate. Let $X = \#$ of trials. Find:

a) $P(X = x)$

b) $P(X > x)$

12. Assume the circuits in each system are all mutually independent. Use the following notation. Let $P(W)$ be the probability that circuit W is working. Find:



a) *P*(System is working)
b) *P*(Circuit C is working given the system is working)
c) *P*(Circuit C is working given the system has failed)

13. Deal 8 cards from a standard deck to 2 persons evenly. So each player gets 4 cards. Find the probability that 1 person has a heart (at least 1) and the other person has none.
Example: The 1^{st} person has ♥♦♣♥ and the 2^{nd} person has ♠♣♦♦.

14. Roll a die 10 times. Find:
a) *P*(exactly 1 number) Example: 1111111111
b) *P*(exactly 2 distinct numbers) Example: 1221222112

15. Draw 10 cards from a standard deck. Find:
a) *P*(exactly 1 suit)
b) *P*(exactly 2 suits)

Chapter 1 Answers

1.1 Equally Likely Outcomes	1	2	3	4	5	6	7	8	9	10
	11	12	13	14						
1.2 Interpretations	1	2	3	4	5	6	7	8	9	
1.3 Distributions	1	2	3	4	5	6	7	8		
1.4 Conditional Probability and Independence	1	2	3	4	5	6	7	8		
1.5 Bayes' Rule	1	2	3	4	5	6	7			
1.6 Sequences of Events	1	2	3	4	5	6	7	8	9	10
	11	12	13	14	15					

1.1 Answers: Equally Likely Outcomes

1.	$P(H) = \frac{1}{2}, P(T) = \frac{1}{2}$		
2. a	$P(1) = P(2) = \cdots = P(6) = \frac{1}{6}$		
b	$P(\text{Prime}) = \frac{3}{6}, \quad P(\text{Composite}) = \frac{2}{6}, \quad P(\text{Neither}) = \frac{1}{6}$		
3. a	$P(S = k) = \begin{cases} \frac{1}{36}(k-1), & k \le 7 \\ \frac{1}{36}(13-k), & k > 7 \end{cases} = \frac{1}{36}(6 - \lvert k-7 \rvert)$		
b	$P(S = k) = \begin{cases} \frac{1}{n^2}(k-1), & k \le n+1 \\ \frac{1}{n^2}(2n+1-k), & k > n+1 \end{cases} = \frac{1}{n^2}(n - \lvert k-n-1 \rvert)$		
4. a	$P(123) = \frac{1}{216}$	b	$P(321) = \frac{1}{216}$
c	$P(1,2,3) = \frac{6}{216}$	d	$P(1,1,1) = \frac{1}{216}$
e	$P(112) = \frac{1}{216}$	f	$P(1,1,2) = \frac{3}{216}$
5. a	$P(A_1A_2B_3) = \frac{30}{216}$	b	$P(A,A,B) = \frac{90}{216}$
c	$P(A,B,C) = \frac{120}{216}$	d	$P(A,A,A) = \frac{6}{216}$
6. a	$7:3 = \frac{7}{3}:1$	b	$3:7 = \frac{3}{7}:1$
7. a	$P(\text{A♠A♥A♦}) = \frac{1}{(52)_3}$	b	$P(\text{A♠, A♥, A♦}) = \frac{6}{(52)_3}$
c	$P(A23) = \frac{64}{(52)_3}$	d	$P(A,2,3) = \frac{384}{(52)_3}$
8. a	$P(S_1S_2S_3) = \frac{52 \cdot 3 \cdot 2}{(52)_3}$	b	$P(A_1B_2C_3) = \frac{52 \cdot 48 \cdot 44}{(52)_3}$
c	$P(2\,S, 1\,D) = 3 \cdot \frac{52 \cdot 3 \cdot 48}{(52)_3}$		
9. a	Not fair	b	Not fair
10. a	$P(U_2 = U_1) = \frac{6}{36}$	b	$P(U_2 > U_1) = \frac{15}{36}$
c	$P(U_2 < U_1) = \frac{15}{36}$	d	$P(\lvert U_2 - U_1 \rvert > 1) = \frac{20}{36}$
e	$P(U_2 - U_1 = k) = \frac{1}{36}(6 - \lvert k \rvert), k = -5, -4, \dots, 4, 5$	f	$P(\lvert U_2 - U_1 \rvert = k) = \begin{cases} \frac{1}{36} \cdot 6, & k = 0 \\ \frac{1}{36}(12 - 2k), & k = 1, \dots, 5 \end{cases}$
11. a	$P(X \ge x) = \frac{1}{36}(7-x)^2$	b	$P(X = x) = \frac{1}{36}(13 - 2x)$
c	$P(Y \le y) = \frac{1}{36}y^2$	d	$P(Y = y) = \frac{1}{36}(2y - 1)$
12. a	$P(X \ge x) = \left[\frac{1}{6}(7-x)\right]^n$	b	$P(X = x) = \left[\frac{1}{6}(7-x)\right]^n - \left[\frac{1}{6}(6-x)\right]^n$
c	$P(Y \le y) = \left(\frac{1}{6}y\right)^n$	d	$P(Y = y) = \left(\frac{1}{6}y\right)^n - \left[\frac{1}{6}(y-1)\right]^n$
13. a	$P(X = n) = \left(\frac{1}{6}\right)^n$	b	$P(X = 0) = \left(\frac{5}{6}\right)^n$
c	$P(X < n) = 1 - \left(\frac{1}{6}\right)^n$	d	$P(X \ge 1) = 1 - \left(\frac{5}{6}\right)^n$

14. a	$P(X = n) = \frac{(13)_{10}}{(52)_{10}}$	b	$P(X = 0) = \frac{(39)_{10}}{(52)_{10}}$
c	$P(X < n) = 1 - \frac{(13)_{10}}{(52)_{10}}$	d	$P(X \geq 1) = 1 - \frac{(39)_{10}}{(52)_{10}}$

1.2 Answers: Interpretations

1.	a	Frequency interpretation	b	Subjective interpretation
	c	Frequency interpretation	d	Subjective interpretation
2.	a	Chance odds against event $= 4164:1$	b	Payoff odds $= 80:1$
	c	Not fair		
3.	a	$P(\text{Quarter Play}) = \frac{4}{38}$	b	$Odds(\text{In Favor}) = 4:34 \approx 0.1176:1$
	c	$Odds(\text{Against}) = 34:4 = 8.5:1$	d	Fair payoff odds $= 8.5:1$
	e	The payoff odds are less than $8.5:1$.		
4.		$P(\text{win the lottery}) < \frac{1}{1{,}000{,}000}$		
5.	a	$P(\text{event assuming fair bet}) = \frac{1}{6}$	b	$ENG = -\frac{352}{1024}$
6.		$ENG = -\frac{2}{11}$		
7.	a	$ENG = \frac{1}{2}$	b	$p = \frac{11}{30}$
8.		$p_1 = \frac{1}{2}, p_2 = \frac{1}{4}, p_3 = \frac{1}{3}, \Sigma = \frac{13}{12}$ If $\Sigma > 1$, then there is not a sure way of winning.		
9.		$p_1 = \frac{1}{2}, p_2 = \frac{1}{4}, p_3 = \frac{1}{5}, \Sigma = \frac{19}{20}$ If $\Sigma < 1$, then there is a sure way of winning.		
	a	See table.	b	See table.
	c			

Horse 1	Horse 2	Horse 3	Win	Net/H1	Net/H2	Net/H3	Net Amt
\$3	0	0	1	3	0	0	3
			2	-3	0	0	-3
			3	-3	0	0	-3
0	\$3	0	1	0	-3	0	-3
			2	0	9	0	9
			3	0	-3	0	-3
0	0	\$3	1	0	0	-3	-3
			2	0	0	-3	-3
			3	0	0	12	12
\$1	\$1	\$1	1	1	-1	-1	-1
			2	-1	3	-1	1
			3	-1	-1	4	2
$\frac{50}{95} \approx 0.5263$	$\frac{25}{95} \approx 0.2632$	$\frac{20}{95} \approx 0.2015$	1	$\frac{50}{95}$	$-\frac{25}{95}$	$-\frac{20}{95}$	$\frac{5}{95}$
			2	$-\frac{50}{95}$	$\frac{75}{95}$	$-\frac{20}{95}$	$\frac{5}{95}$
			3	$-\frac{50}{95}$	$-\frac{25}{95}$	$\frac{80}{95}$	$\frac{5}{95}$

1.3 Answers: Distributions

1.	a	$P(AB) = 0.72$	b	$P(A \cup B) = 0.98$
2.	a	Proof by Venn diagram	b	Algebraic proof
3.		$P(A \cup B \cup C \cup D) = 0.9976$		
4.	a	$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) = 0.80$		
	b	$P(A \cup B \cup C) = \begin{cases} P(ABC) + P(AB\bar{C}) + P(A\bar{B}C) + P(A\bar{B}\bar{C}) \\ \quad +P(\bar{A}BC) + P(\bar{A}B\bar{C}) + P(\bar{A}\bar{B}C) \end{cases} = 0.80$		
	c	$P(A \cup B \cup C) = 1 - P(\bar{A}\bar{B}\bar{C}) = 0.80$		
5.	a	Proof by Venn diagram	b	Proof by Venn diagram
6.	a	$0 \le P(AB) \le 0.32$	b	$0.15 \le P(AB) \le 0.40$
	c	$0.40 \le P(A \cup B) \le 0.72$	d	$0.75 \le P(A \cup B) \le 1$
7.	a	$\sum_{i=1}^{4} P(A_i) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$		
	b	$\sum_{1 \le i < j \le 4} P(A_i A_j) = P(A_1A_2) + P(A_1A_3) + P(A_1A_4) + P(A_2A_3) + P(A_2A_4) + P(A_3A_4)$		
	c	$\sum_{i=1}^{4} P(A_i A_5) = P(A_1A_5) + P(A_2A_5) + P(A_3A_5) + P(A_4A_5)$		
8.	a	$\sum_{i=1}^{n} c = nc$		
	b	$\sum_{1 \le i < j \le 4} P(A_i A_j) + \sum_{i=1}^{4} P(A_i A_5) = \sum_{1 \le i < j \le 5} P(A_i A_j)$		

1.4 Answers: Conditional Probability and Independence

1.	a	A & B are independent. $\Leftrightarrow P(A\|B) = P(A)$		
	b	A & B are positively dependent. $\Leftrightarrow P(A\|B) > P(A)$		
	c	A & B are negatively dependent. $\Leftrightarrow P(A\|B) < P(A)$		
	d	Assume $P(B) \neq 0$. A & B are mutually exclusive. $\Leftrightarrow P(A\|B) = 0$		
2.		$P(A) = 0$ or $P(B) = 0$		
3.	a	$P(S_2) = \frac{1}{4}$	b	$P(S_2\|B_1) = \frac{25}{102} < \frac{1}{4}$
	c	$P(S_2\|C_1) = \frac{13}{51} > \frac{13}{52}$	d	$P(S_2\|F_1) = \frac{1}{4}$
4.		$P(S_2) = \frac{1}{4}$		
5.		$P(A \cup B \cup C) = 1 - 0.59 \cdot 0.52 \cdot 0.49$		
6.	a	$P(rb\|R) = \frac{1}{3}$	b	$P(rr\|R) = \frac{2}{3}$
7.	a	$P(hh\|H_1) = 0.8$	b	H_1 and H_2 are dependent.
	c	H_1 and H_2 are conditionally independent.		
8.	a	$P(hh\|H_1) = \frac{4}{5}$, $\quad P(ht\|H_1) = \frac{1}{5}$	b	$P(H_2\|H_1) = \frac{9}{10}$
	c	$P(hh\|H_1H_2) = \frac{8}{9}$, $\quad P(ht\|H_1H_2) = \frac{1}{9}$	d	$P(H_3\|H_1H_2) = \frac{17}{18}$, $\quad P(H_3\|H_1T_2) = \frac{1}{2}$

1.5 Answers: Bayes' Rule

1.	a	$P(\text{misdiagnosis}) = 0.0103$	b	$P(\text{HIV-}\vert\text{Pos}) = 0.248$
2.	a	$P(F\vert\text{Jack}) > 0.16,\qquad P(F\vert\text{Jack}) = 0.60$	b	$P(\text{Jack}\vert S) = \frac{2}{21}$
	c	$P(\text{Jill}\vert S) = \frac{19}{21}$	d	$P(\text{Jack}\vert F) = \frac{3}{4}$
	e	$P(\text{Jill}\vert F) = \frac{1}{4}$		
3.		$P(L\vert S) = \frac{2}{11}$		
4.	a	$P(B_1B_2\vert B_1) = \frac{1}{2}$	b	$P(B_1B_2\vert X \geq 1) = \frac{1}{3}$
	c	See next parts.	d	$w = \frac{1}{2} \Rightarrow P(B_1B_2\vert X \geq 1, I_B = 1) = \frac{1}{2}$
	e	$w = 1 \Rightarrow P(B_1B_2\vert X \geq 1, I_B = 1) = \frac{1}{3}$	f	$P(B_1B_2\vert X \geq 1, I_B = 1) = \dfrac{1}{1+2w}$
5.	a	$P(B_1B_2\vert A) = \frac{13}{27}$	b	$P(B_1B_2\vert A) = \dfrac{2-\frac{1}{365}}{4-\frac{1}{365}} \approx 0.4997$
	c	$P(B_1B_2\vert A) = \frac{1}{2}$	d	$P(B_1B_2\vert A) = \frac{1}{3}$
6.	a	$P(S\vert X = k) = 1$	b	$P(S\vert X = k) = 0$
	c	$P(S\vert X = k) = \dfrac{k+1}{n+2}$	d	$P(S\vert X = k) = \dfrac{(k+1)p}{n-k+1-(n-2k)p}$
7.		$P(S\vert X = k) = \dfrac{k+1}{n+2}$		

1.6 Answers: Sequences of Events

1.	a	Prove statement.	b	Prove statement.
2.	a	$P(P) = 0.01$	b	$P(\bar{P}S) = 0.99 \cdot 0.01$
	c	$P(\bar{P}\bar{S}L) = 0.99 \cdot 0.98 \cdot 0.05$	d	$P(\bar{P}\bar{S}\bar{L}Z) = 0.99 \cdot 0.98 \cdot 0.95 \cdot 0.07$
	e	$P(\bar{P}\bar{S}\bar{L}\bar{Z}) = 0.99 \cdot 0.98 \cdot 0.95 \cdot 0.93$	f	$P(D) = 1 - 0.99 \cdot 0.98 \cdot 0.95 \cdot 0.93$
	g	$P(D\|\bar{P}\bar{S}) = 1 - 0.95 \cdot 0.93$		
3.	a	Yes to pairwise independent	b	No to mutually independent
4.	a	$p_W = 1 - \left(1 - \frac{1}{365}\right)^n$	b	$p_W \approx 1 - e^{-\frac{1}{365}n}$
	c	$n = 253$	d	$n \approx 253$
5.	a	$P(\bar{D}) = 1 - \frac{(365)_n}{365^n}$	b	$P(\bar{D}) \approx 1 - e^{-\frac{1}{730}n^2}$
	c	$n = 23$	d	$n \approx 23$
6	a	$E(X) \approx -\$0.20$	b	$E(X) \approx \$0.098$
7.	a	$P(W) = 1 - \left(1 - \frac{1}{1000}\right)^n$ $P(W) \approx 1 - e^{-\frac{1}{1000}n}$	b	$n = 693$ $n \approx 694$
	c	$P(\bar{D}) = 1 - \frac{(1000)_n}{1000^n}$ $P(\bar{D}) \approx 1 - e^{-\frac{1}{2000}n^2}$	d	$n = 38$ $n \approx 38$
8.	a	$P(AB) = 0.12, P(AC) = 0.24, P(BC) = 0.18$ $P(AB\bar{C}) = 0.09, P(A\bar{B}C) = 0.21, P(\bar{A}BC) = 0.15, P(\bar{A}\bar{B}\bar{C}) = 0.21$		
	b	Not mutually independent		
9.	a	$n = 5$	b	$P(X = x) = \frac{(12)_{x-1}(x-1)}{12^x}, x = 2, \dots, 13$
10.	a	$P(X = x) = q^{x-1}p$	b	$P(X > x) = q^x$
11.	a	$P(X = x) = \frac{(n)_{x-1}(x-1)}{n^x}, x = 2, \dots, n+1$	b	$P(X > x) = \frac{(n)_x}{n^x}$
12.	a	$P(S) = 0.2471$	b	$P(C\|L) = \frac{0.7 \cdot (1 - 0.1 \cdot 0.2 \cdot 0.4)}{0.9884}$
	c	$P(C\|\bar{L} \cup \bar{E}) = \frac{0.7 \cdot (0.1 \cdot 0.2 \cdot 0.4 \cdot 0.25 + 0.75)}{1 - 0.9884 \cdot 0.25}$		
13.		$P(A) = 2 \cdot \frac{(39)_4}{(52)_4}\left[1 - \frac{(35)_4}{(48)_4}\right]$		
14.	a	$P(X = 1) = 6 \cdot \left(\frac{1}{6}\right)^{10}$	b	$P(X = 2) = \binom{6}{2}\left[\left(\frac{2}{6}\right)^{10} - 2\left(\frac{1}{6}\right)^{10}\right]$
15.	a	$P(X = 1) = 4 \cdot \frac{(13)_{10}}{(52)_{10}}$	b	$P(X = 2) = \binom{4}{2}\left[\frac{(26)_{10}}{(52)_{10}} - 2 \cdot \frac{(13)_{10}}{(52)_{10}}\right]$