Chapter 2: Repeated Trials and Sampling

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2.1	Binomial Distribution
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Chapter 2 Outlines

2.1 Outline: Binomial Distribution

Counting

☐ n factorial

$$n! = \underbrace{n \cdot (n-1) \cdot \ldots \cdot 1}_{n \text{ terms}}$$

Ex: $3! = 6$

☐ n order k

$$(n)_k = \underbrace{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}_{k \text{ terms}} = \frac{n!}{(n-k)!}$$

Ex: $(10)_3 = 10 \cdot 9 \cdot 8 = 720$

☐ n choose k

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, k = 0, 1, \ldots, n$$

Ex: $\binom{10}{3} = \frac{10!}{3!\,7!} = 720$

$$\binom{n}{k} = \frac{(n)_k}{k!}, k = 0, 1, \ldots, n$$

Ex: $\binom{10}{3} = \frac{(10)_3}{3!} = 120$

☐ Values of k such that n choose k evaluates to 0

$$\binom{n}{k} = 0 \text{ if } k \notin \{0, 1, \ldots, n\}$$

Ex: $\binom{10}{12} = 0, \binom{10}{-2} = 0$

☐ Symmetric identity of n choose k

$$\binom{n}{k} = \binom{n}{n-k}$$

Ex: $\binom{10}{3} = \binom{10}{7}$

☐ Special cases of n choose k

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = n$$

☐ n choose $k_1, \ldots, k_r$

Suppose $k_1 + \cdots + k_r = n$.

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!\,k_2! \cdot \ldots \cdot k_r},$$

$$\binom{n}{k_1, k_2, \ldots, k_r} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3} \cdot \ldots \cdot \binom{n-k_1-\cdots-k_{r-1}}{k_r}$$

Ex: $\binom{10}{2, 3, 1, 4} = \frac{10!}{2!\,3!\,1!\,4!} = 12600$

Ex: $\binom{10}{2, 3, 1, 4} = \binom{10}{2}\binom{8}{3}\binom{5}{1}\binom{4}{4} = 12600$

Pascal's Triangle (tilted)

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=0$	$\binom{0}{0}=1$					
$n=1$	$\binom{1}{0}=1$	$\binom{1}{1}=1$				
$n=2$	$\binom{2}{0}=1$	$\binom{2}{1}=2$	$\binom{2}{2}=1$			
$n=3$	$\binom{3}{0}=1$	$\binom{3}{1}=3$	$\binom{3}{2}=3$	$\binom{3}{0}=1$		
$n=4$	$\binom{4}{0}=1$	$\binom{4}{1}=4$	$\binom{4}{2}=6$	$\binom{4}{3}=4$	$\binom{4}{0}=1$	
$n=5$	$\binom{5}{0}=1$	$\binom{5}{1}=5$	$\binom{5}{2}=10$	$\binom{5}{3}=10$	$\binom{5}{4}=5$	$\binom{5}{0}=1$

Relationships of Consecutive Binomial Coefficients

Relationships	Let $\binom{n}{k}=\binom{4}{2}$.
$\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$	$\binom{4}{2}+\binom{4}{3}=\binom{5}{3}$
$\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$	$\binom{4}{1}+\binom{4}{2}=\binom{5}{2}$
$\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}$	$\binom{3}{1}+\binom{3}{2}=\binom{4}{2}$

Interpretation

Symbol	Expression	Number
$n!$	n factorial	# of ways to order n distinct elements
$(n)_k$	n order k	# of ways to order n distinct elements in k positions
$\binom{n}{k}$	n choose k	# of ways to choose k unordered elements from n distinct elements # of ways to order n elements where you have k elements of one type and $n-k$ elements of another type
$\binom{n}{k_1, k_2, \dots, k_r}$,	n choose $k_1, k_2, \dots, k_r$	# of ways to choose $k_1, k_2, \dots, k_r$ unordered elements from n distinct elements # of ways to order n elements where there are r types and k_i elements of type i

Permutations

$1! = 1$

1	A

$2! = 2$

1	AB
2	BA

$3! = 6$

1	ABC
2	ACB
3	BAC
4	BCA
5	CAB
6	CBA

$4! = 24$

1	ABCD	7	BACD	13	CABD	19	DABC
2	ABDC	8	BADC	14	CADB	20	DACB
3	ACBD	9	BCAD	15	CBAD	21	DBAC
4	ACDB	10	BCDA	16	CBDA	22	DBCA
5	ADBC	11	BDAC	17	CDAB	23	DCAB
6	ADCB	12	BDCA	18	CDBA	24	DCBA

Combinations and Number of Arrangements

	A	**B**	**C**	**D**	**E**	**1**	**2**	**3**	**4**	**5**
1						0	0	0	0	0

$$\binom{5}{0} = \frac{5!}{0! \cdot 5!} = 1$$

	A	**B**	**C**	**D**	**E**	**1**	**2**	**3**	**4**	**5**
1	A					1	0	0	0	0
2		B				0	1	0	0	0
3			C			0	0	1	0	0
4				D		0	0	0	1	0
5					E	0	0	0	0	1

$$\binom{5}{1} = \frac{5!}{1! \cdot 4!} = 5$$

	A	**B**	**C**	**D**	**E**	**1**	**2**	**3**	**4**	**5**
1	A	B				1	1	0	0	0
2	A		C			1	0	1	0	0
3	A			D		1	0	0	1	0
4	A				E	1	0	0	0	1
5		B	C			0	1	1	0	0
6		B		D		0	1	0	1	0
7		B			E	0	1	0	0	1
8			C	D		0	0	1	1	0
9			C		E	0	0	1	0	1
10				D	E	0	0	0	1	1

$$\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$$

	A	**B**	**C**	**D**	**E**	**1**	**2**	**3**	**4**	**5**
1	A	B	C			1	1	1	0	0
2	A	B		D		1	1	0	1	0
3	A	B			E	1	1	0	0	1
4	A		C	D		1	0	1	1	0
5	A		C		E	1	0	1	0	1
6	A			D	E	1	0	0	1	1
7		B	C	D		0	1	1	1	0
8		B	C		E	0	1	1	0	1
9		B		D	E	0	1	0	1	1
10			C	D	E	0	0	1	1	1

$$\binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10$$

	A	**B**	**C**	**D**	**E**	**1**	**2**	**3**	**4**	**5**
1	A	B	C	D		1	1	1	1	0
2	A	B	C		E	1	1	1	0	1
3	A	B		D	E	1	1	0	1	1
4	A		C	D	E	1	0	1	1	1
5		B	C	D	E	0	1	1	1	1

$$\binom{5}{4} = \frac{5!}{4! \cdot 1!} = 5$$

	A	**B**	**C**	**D**	**E**	**1**	**2**	**3**	**4**	**5**
1	A	B	C	D	E	1	1	1	1	1

$$\binom{5}{5} = \frac{5!}{0! \cdot 5!} = 1$$

Binomial Distribution

- Binomial Formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$
$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

- Binomial Distribution

Let $X = \#$ of successes in n independent trials, where $p =$ probability of success for each trial.

$X \sim Bin(n,p)$ on $\{0, 1, \ldots, n\}$

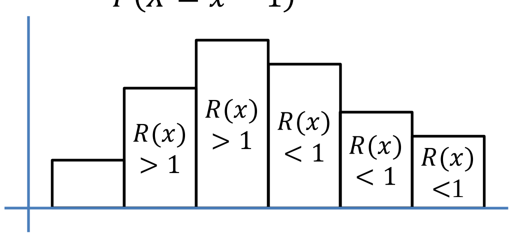
$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

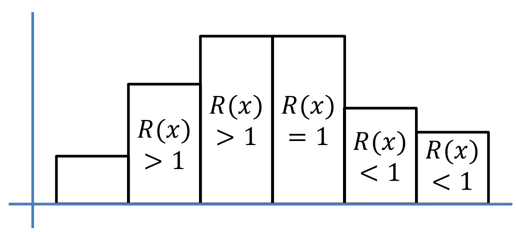
- Conditions to use the Binomial Distribution

i) n is fixed.
ii) p is constant.
iii) Trials are independent.

- Consecutive Odds Ratio

$$R(x) = \frac{P(X = x)}{P(X = x-1)}$$





To find the mode, look for the largest x such that $R(x) \geq 1$.

- Binomial Consecutive Odds Ratio

$$R(x) = \frac{P(X = x)}{P(X = x-1)} = \frac{n-x+1}{x} \cdot \frac{p}{q}, \qquad x = 1, 2, \ldots, n$$

- Mode

Let $m = \lfloor np + p \rfloor$.

$$\text{Mode}(X) = \begin{cases} m, & np + p \notin \mathbb{Z}^+ \\ m, m-1, & np + p \in \mathbb{Z}^+ \end{cases}$$

- Multinomial Distribution: Extension of the Binomial Distribution

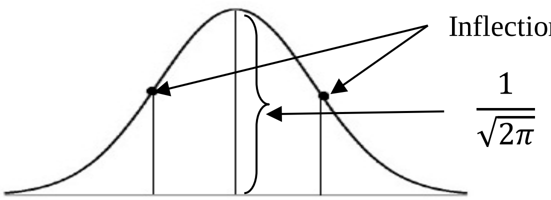
$\mathbf{X} \sim Multin(n, p_1, p_2, \ldots, p_k)$

$$P(\mathbf{X} = \mathbf{x}) = \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \cdot \ldots \cdot p_k^{x_k}$$

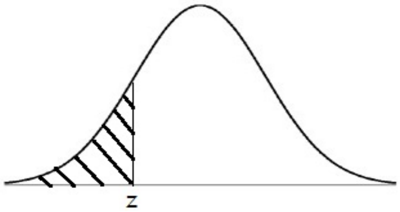
where $\begin{cases} \mathbf{x} = (x_1, x_2, \ldots, x_k) \\ x_1 + x_2 + \cdots + x_k = n \end{cases}$

2.2 Outline: Normal Approximation: Method

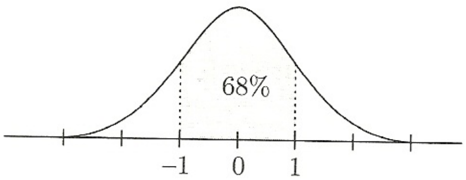
- Standard Normal
  $Z \sim N(0,1)$
  $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$
  The inflections are at: $z = \pm 1$
  Symmetry: $\phi(-z) = \phi(z)$

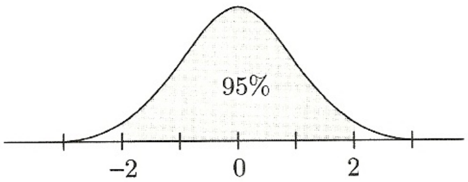


- Cumulative Standard Normal
  $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$



- Middle Area of the Standard Normal
  $P(-1 < Z < 1) \approx 68\%$, $P(-2 < Z < 2) \approx 95\%$, $P(-3 < Z < 3) \approx 99.7\%$





  Comment: More accurately, $P(-2 < Z < 2) \approx 95.4\%$ and $P(-1.96 < Z < 1.96) \approx 95\%$

- Areas of a Standard Normal Curve
  $P(Z < z) = \Phi(z)$
  $P(Z > z) = 1 - \Phi(z)$
  $P(a < Z < b) = \Phi(b) - \Phi(a)$
  $P(-z < Z < z) = 2\Phi(z) - 1$

- Normal
  $X \sim N(\mu, \sigma^2)$
  $\phi(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- Square Root Law for Binomial
  Suppose $X \sim Bin(n,p)$.
  $SD(X) = \sqrt{np(1-p)}$
  Let $\hat{p} = \frac{1}{n}X$.
  $$SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

- Normal Approximation to the Binomial
  $X \sim Bin(n,p)$ on $\{0, 1, \ldots, n\}$, $\mu = E(X) = np$, $\sigma = SD(X) = \sqrt{np(1-p)}$
  $$P(a \le X \le b) \approx \Phi\left(\frac{b+\frac{1}{2}-\mu}{\sigma}\right) - \Phi\left(\frac{a-\frac{1}{2}-\mu}{\sigma}\right)$$

- Conditions when the Normal Approximation to the Binomial is appropriate
  Large σ: $\sqrt{npq} \ge 3$
  As n increases, then σ increases.
  As p gets closer to 0.5 from either direction (less skewed), then σ increases.

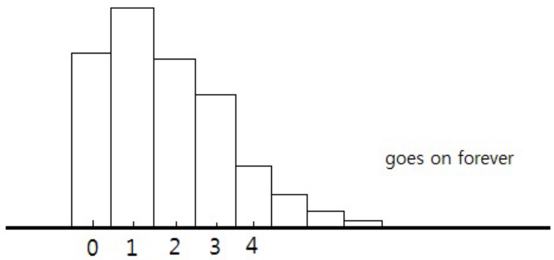
- Law of Large Numbers (Weak Law of Numbers)
  $\forall \varepsilon > 0, \lim_{n\to\infty} P(|\hat{p} - p| < \varepsilon) = 1$

- Higher Orders of the Standard Normal
  $\phi'(z) = -z\phi(z)$ (2.2-15a)
  $\phi''(z) = (z^2 - 1)\phi(z)$ (2.2-15b)
  $\phi'''(z) = (-z^3 + 3z)\phi(z)$ (2.2-16a)

2.4 Outline: Poisson Approximation

- [ ] Taylor Series of e^x
  $e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots$

- [ ] Poisson Distribution
  $X \sim Pois(\mu)$ on $\{0, 1, 2, \ldots\}$
  $P(X = x) = e^{-\mu}\dfrac{\mu^x}{x!}$
  $P(X = 0) = e^{-\mu}$
  $P(X \geq 1) = 1 - e^{-\mu}$



  Comment: The range of a Binomial distribution is $x = 0, 1, 2, \ldots, n$. In a Poisson, the range is from 0 to ∞. For a small μ, the Poisson probabilities are stacked at the lower values of x. This is a skewed right distribution.

- [ ] Poisson Approximation to the Binomial (Guidelines)
  Ideal for large n and small p where $\mu = np \leq 3$
  However, if n is too large and p is not small enough such that $\mu = np > 9$, then use a normal approximation.

- [ ] Consecutive Odds Ratio
  $R(x) = \dfrac{P(X = x)}{P(X = x - 1)} = \dfrac{\mu}{x}, x = 1, 2, \ldots$

- [ ] Mode
  Let $m = \lfloor \mu \rfloor$.
  $\text{Mode}(X) = \begin{cases} m, & m \notin \mathbb{Z}^+ \\ m, m-1, & m \in \mathbb{Z}^+ \end{cases}$

- [ ] Counting the Complement
  If n is large and p is large, count the complement event.

2.5 Outline: Random Sampling

- Hypergeometric Distribution

$X \sim Hyperg(n, N, G)$ on $\{\max(0, n+G-N), \dots, \min(n, G)\}$

$$P(X=x) = \frac{\binom{G}{x}\binom{N-G}{n-x}}{\binom{N}{n}}$$

$$\begin{cases} G-x \geq 0 \\ n-x \geq 0 \end{cases} \Rightarrow \begin{cases} x \leq G \\ x \leq n \end{cases} \Rightarrow x \leq \min(n, G)$$

$$\begin{cases} x \geq 0 \\ N-n-G+x \geq 0 \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ x \geq n+G-N \end{cases} \Rightarrow x \geq \max(0, n+G-N)$$

- Relating the Hypergeometric Probability to a Sequence of Probabilities

$$P(X=x) = \frac{\binom{G}{g}\binom{B}{b}}{\binom{N}{n}} = \binom{n}{g}\frac{(G)_g (B)_b}{(N)_n}$$

where $G + B = N, g + b = n$

- Calculating the Hypergeometric probability from either direction in a 2 by 2 table.

	Good	Bad	
Sample	x	$n-x$	n
Urn	$G-x$	$B-n+x$ or $N-n-G+x$	$N-n$
	G	B	N

In a sample of n, we want x good elements and $n-x$ bad elements.

$$P(X=x) = \frac{\binom{G}{g}\binom{B}{b}}{\binom{N}{n}}$$

Of the G good elements, assign x to the sample and $G-x$ to the urn.

$$P(X=x) = \frac{\binom{n}{x}\binom{N-n}{G-x}}{\binom{N}{G}}$$

- Extension of the Hypergeometric Distribution

$$P(\mathbf{X}=\mathbf{x}) = \frac{\binom{N_1}{x_1}\binom{N_2}{x_2}\cdot \dots \cdot \binom{N_k}{x_k}}{\binom{N}{n}}$$

where $\begin{cases} \mathbf{x} = (x_1, x_2, \dots, x_k) \\ x_1 + x_2 + \cdots + x_k = n \\ N_1 + N_2 + \cdots + N_k = N \end{cases}$

Chapter 2 Worksheets

2.1 Binomial Distribution	1	2	3	4	5	6	7	8	9	10
	11	12								
2.2 Normal Approximation: Method	1	2	3	4	5	6				
2.4 Poisson Approximation	1	2	3	4						
2.5 Random Sampling	1	2	3	4	5	6	7	8	9	10
	11									

2.1 Worksheet: Binomial Distribution

1. How many ways are there to arrange the following elements?
a) ABCDE
b) AABBB
c) ABCDEFGHIJ
d) AABBBCCCCD

2. Out of 10 students, how many ways are there to count the following?
a) Select 3 students as officers.
b) Select 3 students as officers where there is a president, a vice-president, and a treasurer.
c) Assign numerical scores from $1, 2, \ldots, 10$ to all 10 students and there are no ties.
d) Assign grades of 2 A's, 3 B's, 3 C's, 1 D and 1 F.

3. Out of 10 people, how many ways are there to choose the following? Assume all groups formed are *distinguishable*.
a) Form a group of 3.
b) Form 2 groups of 5.
c) Form 2 groups of 3, a group of 2, and 2 groups of 1.

4. Out of 32 people, how many ways are to form *indistinguishable* groups?
a) Form a group of 15 and a group of 17.
b) Form 2 groups of 16 each.
c) Form 3 groups of 2, 4 groups of 5, and a group of 6.

5. Toss a p-coin 5 times and record the outcome of each trial. Find the probability of the following events.
a) HHHTT
b) HTHTH
c) 3 heads
d) At least 1 head
e) Not all heads
f) An even number of heads

6. Roll a fair die 10 times. Let X be the number of ones. Find:
a) average of X
b) mode of X
c) modes of X if the die is rolled 11 times

7. Suppose $X \sim Bin(n, p)$.
a) Prove the mean of a binomial distribution.
b) Prove the consecutive odds ratio (or ratio of consecutive probabilities) of a binomial distribution.
c) Prove the mode of a binomial distribution.

8. Simplify each of the following summation expressions. Also match each summation to a problem that involves rolling a die 10 times.

a) $\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

b) $\sum_{x=1}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

c) $\sum_{x=0}^{n} \binom{n}{x} p_1^x p_2^{n-x}$

d) $\sum_{x=1}^{n-1} \binom{n}{x} p_1^x p_2^{n-x}$

A. Find the probability that you get at least 1 six.
B. Find the probability that you get ones and twos and no other values in your 10 trials.
C. Find the probability that you get either a one or a two on each trial.
D. Find the probability that you get 0 to 10 sixes.

9. A license plate is made of 3 letters and 5 numbers. A letter or number can be used more than once unless the numbers and letters are specified. Find the number of license plates that can be created under each condition.

a) given 8 numbers and letters: *ABC12345*
b) given 8 numbers and letters: *AAB11122*
c) 3 given letters, *ABC*, followed by the 5 numbers, *12345*
d) 3 given letters, *AAB*, followed by the 5 numbers, *11123*
e) 3 letters followed by 5 numbers
f) 3 unique letters followed by 5 unique numbers
g) 3 unique letters in alphabetical order followed by 5 unique numbers in ascending order
h) 3 letters and 5 numbers (Ex – *S777E86E*)
i) 3 unique letters and 5 unique numbers (Ex – *S123L98C*)
j) 3 unique letters in alphabetical order and 5 unique numbers in ascending order (Ex – *A135C68E*)

10. Roll a die 20 times. Let $X =$ # of sixes in the 20 trials. Let $X_1 =$ # of sixes in the first 5 trials, $X_2 =$ # of sixes in the last 15 trials, and $X_3 =$ # of sixes in the last 19 trials. Find:

a) P(4 sixes)
b) P(4 sixes | 1^{st} trial is a six)
c) P(4 sixes | at least 1 six)
d) P(2 sixes in the first 5 trials | 1^{st} five showed up on the 6^{th} trial)
e) P(4 sixes | at least 1 five)
f) P(1 six in the first 5 trials and 3 sixes in the last 15 trials)
g) P(1 six in the first 5 trials | 4 sixes)
h) P(more than 3 sixes in the 1^{st} 5 trials and more than 5 sixes total)

11. Roll a die until you see 3 sixes. Let $X =$ # of trials when you stop.

a) $P(X = 10)$
b) $P(X > 10)$

12. Roll a fair die 12 times. Let X be the number of ones, Y be the number of twos or threes, and Z be the number of fours, fives, or sixes. Find:

a) P(2 ones, 3 twos or threes, 7 fours, fives, or sixes)
b) P(2 ones given there are no twos and no threes)

2.2 Worksheet: Normal Approximation: Method

1. Suppose $Z \sim N(0,1)$. Given $\Phi(1.28) = 0.90$, find the following:
a) $\Phi(-1.28)$
b) $P(-1 < Z < 2)$

2. Roll a die 600 times. Let $X = \#$ of sixes. Approximate the following probabilities with a normal distribution.
a) P(more than 120 sixes)
b) P(more than 20% sixes)
c) P(between 90 and 115 sixes inclusive)

3. A student guesses on all 100 questions of a multiple choice exam. There are 5 choices per question, only 1 of which is correct. Let $X = \#$ of questions that he guesses correctly. Use the following methods to find the probability he gets 20 correct.
a) Binomial
b) Cumulative density function (CDF) of normal curve
c) Normal curve density function
d) Stirling's Approximation to the Binomial (optional)

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

4. A student guesses on all 100 questions of a multiple choice exam. There are 5 choices per question, only 1 of which is correct. Suppose the student gets 4 points for each correct answer and loses 1 point for each wrong answer. Let $Y = \#$ of total points. Use the following methods to find the probability he gets 0 points.
a) Cumulative density function (CDF) of normal curve
b) Normal curve density function

5. Roll a die 600 times. Let $X = \#$ of sixes. The probability of getting $100 \pm d$ sixes is approximately 90%. Find d.

6. Roll a die until you see 100 sixes. Let $X = \#$ of trials when you stop. Find the probability exactly using a computer program, and approximate with the normal distribution.
a) $P(X = 613)$
b) $P(X > 612)$

2.4 Worksheet: Poisson Approximation

1. Derive the following properties of the Poisson distribution.
a) Prove the sum of all the probabilities over its range is 1.

$$\sum_{x=0}^{\infty} e^{-\mu}\frac{\mu^x}{x!} = 1$$

b) Derive the Poisson distribution using the limit definition of e.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

c) Derive the Poisson distribution using consecutive odds ratio.

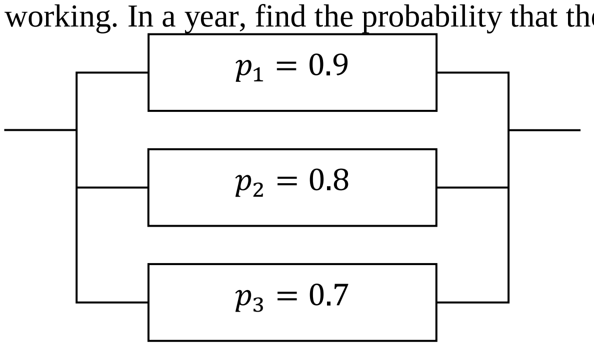
$$R(x) = \frac{P(x)}{P(x-1)} = \frac{\mu}{x}, x = 1, 2, \ldots$$

2. In a statistics class of 1000 students, each student will flip a coin 10 times. Find the following probabilities exactly using the binomial distribution and approximately using the Poisson distribution.
a) P(no one will get all heads)
b) P(at least 1 student will get all heads)
c) P(3 students will get all heads)
d) P(less than 3 students will get all heads)

3. Prove the mode of a Poisson distribution has the following properties. There is either a single mode or a double mode. A double mode occurs when μ is an integer.
Let $m = \lfloor \mu \rfloor$.

$$\text{Mode}(X) = \begin{cases} m, & m \notin \mathbb{Z}^+ \\ m, m-1, & m \in \mathbb{Z}^+ \end{cases}$$

4. Suppose that every day that this circuit system is turned on is considered an independent trial. Also assume the circuits are independent. Let p_i be the probability that the i^{th} circuit is working. In a year, find the probability that the system is working for more than 361 days.



2.5 Worksheet: Random Sampling

1. How many ways are there to choose 3 elements from {ABCDEFGHIJ} under each condition? (The order does not matter.)

a) An element can be chosen once.
b) An element can be chosen more than once. There are 3 of a kind.
c) An element can be chosen more than once. There are 1 of a kind and 2 of a kind.
d) An element can be chosen more than once. There are 3 distinct elements (same as part a)).
e) An element can be chosen more than once.

2. Draw cards without replacement from a standard deck. Find the probability distribution of X and state its range.

a) Draw 5 cards from a deck. Let $X = \#$ of hearts.
b) Draw 5 cards from a deck. Let $X = \#$ of aces.
c) Draw 20 cards from a deck. Let $X = \#$ of numeric cards. A numeric card is defined to be any card with a rank from an ace through ten.
d) Draw 40 cards from a deck. Let $X = \#$ of hearts.

3. In an urn of 40 green and 60 blue marbles, draw 8 marbles without replacement.

a) Place the appropriate numbers into a 2 by 2 table.

	Green	Blue	
Sample	x	$n - x$	n
Urn	$G - x$	$N - n - G + x$	$N - n$
	G	B	N

b) Find the probability of getting 3 green marbles.

4. In an urn of 40 green, 60 blue, and 50 red marbles, draw 12 marbles without replacement.

a) Place the appropriate numbers into a 2 by 3 table.

	Green	Blue	Red	
Sample	x	y	z	n
Urn	$G - x$	$B - y$	$R - z$	$N - n$
	G	B	R	N

b) Find the probability of getting 3 green, 5 blue, and 4 red marbles.

5. Deal n_i cards from a standard deck to 4 players. Let $X_i = \#$ of hearts that the i^{th} player gets. Find:

	Player 1	Player 2	Player 3	Player 4	
Hearts	$x_1 = 2$	$x_2 = 3$	$x_3 = 1$	$x_4 = 7$	$G = 13$
Non-Hearts	$n_1 - x_1 = 8$	$n_2 - x_2 = 17$	$n_3 - x_3 = 5$	$n_4 - x_4 = 9$	$B = 39$
	$n_1 = 10$	$n_2 = 20$	$n_3 = 6$	$n_4 = 16$	$N = 52$

a) $P(X_1 = 2)$
b) $P(X_1 = 2, X_2 = 3, X_3 = 1, X_4 = 7)$

6. Out of a total of $N = 52$ raffle tickets, each of 4 players buys 13 tickets. There are 3 prizes. Find:
a) P(1 triple winner)
b) P(1 single winner and 1 double winner)
c) P(3 single winners)

7. There are 4 winning tickets out of 200. Distribute the 200 tickets evenly to 10 individuals. Let $X = \#$ of individuals that have won.
a) Find the distribution of X.
b) If there are exactly 2 winners, find the probability they have the same number of winning tickets.

8. Draw $n = 10$ cards from a deck. Find the probability of:
a) no large cards {Large cards: 10, J, Q, K, A};
b) at least one ace, but no other large cards;
c) no aces, but at least 1 other large card;
d) at most one kind of large card.

9. In a poker hand of $n = 5$ cards, find the following probabilities. (Deal 5 cards without replacement.)
a) P(3 of a kind, 2 of a kind) {ranks: 3a, 2b}
b) P(2 pairs) {ranks: 2a, 2b, c}

10. In a poker hand of $n = 7$ cards, find the following probabilities.
a) P(4 of a kind, 3 of a kind) {ranks: 4a, 3b}
b) P(4 of a kind, a pair) {ranks: 4a, 2b, c}
c) P(3 of a kind, 2 pairs) {ranks: 3a, 2b, 2c}
d) P(2 pairs) {ranks: 2a, 2b, c, d, e}

11. Deal 13 cards each from a standard deck to 4 players. Find the probability of each event:
a) each player has an ace
b) 1 player has a pair of aces, and 2 other players have 1 each
c) exactly 2 players have a pair of aces
d) 1 player has 3 aces and another player has 1 ace

Chapter 2 Answers

2.1 Binomial Distribution	1	2	3	4	5	6	7	8	9	10
	11	12								
2.2 Normal Approximation: Method	1	2	3	4	5	6				
2.4 Poisson Approximation	1	2	3	4						
2.5 Random Sampling	1	2	3	4	5	6	7	8	9	10
	11									

2.1 Answers: Binomial Distribution

1.	a	$\# = 5!$	b	$\# = \frac{5!}{2!\,3!} = \binom{5}{2} = \binom{5}{3}$
	c	$\# = 10!$	d	$\# = \binom{10}{2,3,4,1}$
2.	a	$\# = \binom{10}{3}$	b	$\# = (10)_3$
	c	$\# = 10!$	d	$\# = \binom{10}{2,3,3,1,1}$
3.	a	$\# = \binom{10}{3}$	b	$\# = \binom{10}{5}$
	c	$\# = \binom{10}{3,3,2,1,1}$		
4.	a	$\# = 1 \cdot \binom{32}{15}$	b	$\# = \frac{1}{2!} \cdot \binom{32}{16}$
	c	$\# = \frac{1}{3!\,4!} \cdot \binom{32}{2,2,2,5,5,5,5,6}$		
5.	a	$P(\text{HHHTT}) = p^3q^2$	b	$P(\text{HTHTH}) = p^3q^2$
	c	$P(X=3) = \binom{5}{3}p^3q^2$	d	$P(X \geq 1) = 1 - q^5$
	e	$P(X \neq 5) = 1 - p^5$	f	$P(X \in \{2,4,6\})$ $= \binom{5}{0}q^5 + \binom{5}{2}p^2q^3 + \binom{5}{4}p^4q$
6.	a	$\mu = \frac{10}{6}$	b	$\text{mode}(X) = 1$
	c	$\text{mode}(X) = 2, 1$		
7.	a	$\mu = np$ ■	b	$R(x) = \frac{n-x+1}{x} \cdot \frac{p}{q}$ ■
	c	$\text{Mode}(X) = \begin{cases} m, & np+p \notin \mathbb{Z}^+ \\ m, m-1, & np+p \in \mathbb{Z}^+ \end{cases}$ ■		
8.	a	$\sum_{x=0}^{n} \binom{n}{x} p^x(1-p)^{n-x} = 1$, D	b	$\sum_{x=1}^{n} \binom{n}{x} p^x(1-p)^{n-x} = 1 - q^n$, A
	c	$\sum_{x=0}^{n} \binom{n}{x} p_1^x p_2^{n-x}$ $= (p_1+p_2)^n$, C	d	$\sum_{x=1}^{n-1} \binom{n}{x} p_1^x p_2^{n-x}$ $= (p_1+p_2)^n - p_1^n - p_2^n$, B
9.	a	$\# = 8!$	b	$\# = \binom{8}{2,1,3,2} = \frac{8!}{2!\,3!\,2!}$
	c	$\# = 3! \cdot 5!$	d	$\# = \frac{3!}{2!} \cdot \frac{5!}{3!}$
	e	$\# = 26^3 \cdot 10^5$	f	$\# = (26)_3 \cdot (10)_5$
	g	$\# = \binom{26}{3} \cdot \binom{10}{5}$	h	$\# = \frac{8!}{3!\,5!} \cdot 26^3 \cdot 10^5$
	i	$\# = \frac{8!}{3!\,5!} \cdot (26)_3 \cdot (10)_5 \quad = 8! \cdot \binom{26}{3} \cdot \binom{10}{5}$	j	$\# = \frac{8!}{3!\,5!} \cdot \binom{26}{3} \cdot \binom{10}{5}$

10. a	$P(X=4)=\binom{20}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{16}$	b	$P(X=4\mid I_1=1)=\binom{19}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{16}$
c	$P(X=4\mid X\geq 1)=\dfrac{\binom{20}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{16}}{1-\left(\frac{5}{6}\right)^{20}}$	d	$P(X_1=2\mid W=6)=\binom{5}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3$
e	$P(X=4\mid T_1\geq 1)$ $=\dfrac{\binom{20}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{16}\cdot\left[1-\left(\frac{4}{5}\right)^{16}\right]}{1-\left(\frac{5}{6}\right)^{20}}$	f	$P(X_1=1,X_2=3)$ $=\binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4\cdot\binom{15}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{12}$
g	$P(X_1=1\mid X=4)=\dfrac{\binom{5}{1}\binom{15}{3}}{\binom{20}{4}}$		
h	$P(X_1\geq 4,X\geq 6)=\binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1\left[1-\left(\frac{5}{6}\right)^{15}-\binom{15}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{14}\right]+\left(\frac{1}{6}\right)^5\left[1-\left(\frac{5}{6}\right)^{15}\right]$		
11. a	$P(X=10)=\binom{9}{2}\left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)^3$		
b	$P(X>10)=\left(\frac{5}{6}\right)^{10}+\binom{10}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^9+\binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$		
12. a	$P(X=2,Y=3,Z=7)=\binom{12}{2,3,7}\left(\frac{1}{6}\right)^2\left(\frac{2}{6}\right)^3\left(\frac{3}{6}\right)^7$		
b	$P(X=2\mid Y=0)=\binom{12}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{10}$		

2.2 Answers: Normal Approximation: Method

1.	a	$\Phi(-z) = 0.10$	b	$P(-1 < Z < 2) = \Phi(2) - \Phi(-1) \approx 0.815$
2.	a	$P(X > 120) \approx 1 - \Phi(2.246) \quad \approx 0.01235$	b	$P(\hat{p} > 0.2) \approx 1 - \Phi(2.246) \quad \approx 0.01235$
	c	$P(90 \le X \le 115) \approx \Phi(1.70) - \Phi(-1.15) \quad \approx 0.8302$		
3.	a	$P(X = 20) = \binom{100}{20}\left(\frac{1}{5}\right)^{20}\left(\frac{4}{5}\right)^{80} \quad \approx 0.099300$		
	b	$P(X = 20) \approx 2\Phi\left(\frac{1}{8}\right) - 1 \quad \approx 0.099477$		
	c	$P(X = 20) \approx \dfrac{1}{\sqrt{2\pi} \cdot 4} \quad \approx 0.099736$		
	d	$P(X = 20) \approx \dfrac{1}{\sqrt{2\pi} \cdot 4} \quad \approx 0.099736$		
4.	a	$P(Y = 0) \approx 2\Phi\left(\frac{1}{8}\right) - 1 \quad \approx 0.099477$	b	$P(Y = 0) \approx \dfrac{1}{\sqrt{2\pi} \cdot 4} \quad \approx 0.099736$
5.		$d = 15$		
6.	a	$P(X = 613) = \binom{612}{99}\left(\frac{1}{6}\right)^{99}\left(\frac{5}{6}\right)^{513} \cdot \frac{1}{6} \quad \approx 0.006914$ $P(X = 613) \approx \left[\Phi\left(-\frac{2.5}{\sqrt{85}}\right) - \Phi\left(-\frac{3.5}{\sqrt{85}}\right)\right] \cdot \frac{1}{6} \quad \approx 0.006837$		
	b	$P(X > 612) = \sum_{y=0}^{99} \binom{612}{y}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{612-y} \quad \approx 0.3975$ $P(X > 612) \approx \Phi\left(-\frac{2.5}{\sqrt{85}}\right) \quad = 0.3931$		

2.4 Answers: Poisson Approximation

1. a	$\sum_{x=0}^{\infty} e^{-\mu}\frac{\mu^x}{x!} = 1$ ■	b	$P(X=x) = \frac{\mu^x}{x!}e^{-\mu}$ ■	
c	$R(x) = \frac{P(x)}{P(x-1)} = \frac{\mu}{x}, x = 1, 2, \ldots$ ■			
2. a	$P(X=0) = \left(\frac{1023}{1024}\right)^{1000} \approx 0.3677$ $P(Y=0) = e^{-1000/1024} \approx 0.3766$	b	$P(X \geq 1) = 1 - \left(\frac{1023}{1024}\right)^{1000} \approx 0.6323$ $P(Y \geq 1) = 1 - e^{-1000/1024} \approx 0.6234$	
c	$P(X=3) = \binom{1000}{3}\left(\frac{1}{1024}\right)^3\left(\frac{1023}{1024}\right)^{997} \approx 0.0571$ $P(Y=3) = e^{-1000/1024} \cdot \frac{(1000/1024)^3}{3!} \approx 0.0585$			
d	$P(X<3) = \sum_{i=0}^{2}\binom{1000}{i}\left(\frac{1}{1024}\right)^i\left(\frac{1023}{1024}\right)^{1000-i} \approx 0.9026$ $P(Y<3) = \sum_{i=0}^{2} e^{-1000/1024}\frac{(1000/1024)^i}{i!} \approx 0.9240$			
3	Let $m = \lfloor \mu \rfloor$. $\text{Mode}(X) = \begin{cases} m, & m \notin \mathbb{Z}^+ \\ m, m-1, & m \in \mathbb{Z}^+ \end{cases}$			
4	$P(X > 361) \approx e^{-2.19}\left(1 + \frac{2.19}{1!} + \frac{2.19^2}{2!} + \frac{2.19^3}{3!}\right)$			

2.5 Answers: Random Sampling

<table>
<tr><td>1.</td><td>a</td><td>$\# = 120$</td><td>b</td><td>$\# = 10$</td></tr>
<tr><td></td><td>c</td><td>$\# = 90$</td><td>d</td><td>$\# = 120$</td></tr>
<tr><td></td><td>e</td><td>$\# = 220$</td><td></td><td></td></tr>
<tr><td>2.</td><td>a</td><td>$P(X = x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}, x = 0, 1, \dots, 5$</td><td>b</td><td>$P(X = x) = \frac{\binom{4}{x}\binom{48}{5-x}}{\binom{52}{5}}, x = 0, 1, \dots, 4$</td></tr>
<tr><td></td><td>c</td><td>$P(X = x) = \frac{\binom{40}{x}\binom{12}{20-x}}{\binom{52}{20}}, x = 8, 9, \dots, 20$</td><td>d</td><td>$P(X = x) = \frac{\binom{13}{x}\binom{39}{40-x}}{\binom{52}{40}}, x = 1, 2, \dots, 13$</td></tr>
<tr><td>3.</td><td>a</td><td colspan="3">

	Green	Blue	
Sample	$x = 3$	$n - x = 5$	$n = 8$
Urn	$G - x = 37$	$N - n - G + x = 55$	$N - n = 92$
	$G = 40$	$B = 60$	$N = 100$

</td></tr>
<tr><td>3.</td><td>b</td><td colspan="3">$P(X = 3) = \frac{\binom{40}{3}\binom{60}{5}}{\binom{100}{8}}$</td></tr>
<tr><td>4.</td><td>a</td><td colspan="3">

	Green	Blue	Red	
Sample	x	y	z	$n = 12$
Urn	$G - x = 40 - x$	$B - y = 60 - y$	$R - z = 50 - z$	$N - n = 138$
	$G = 40$	$B = 60$	$R = 50$	$N = 150$

</td></tr>
<tr><td>4.</td><td>b</td><td colspan="3">$P(X = 3, Y = 5, Z = 4) = \frac{\binom{12}{3}\binom{138}{37}}{\binom{150}{40}} \cdot \frac{\binom{9}{5}\binom{101}{55}}{\binom{110}{60}} \cdot \frac{\binom{4}{4}\binom{46}{46}}{\binom{50}{50}} = \frac{\binom{40}{3}\binom{60}{5}\binom{50}{4}}{\binom{150}{12}}$</td></tr>
<tr><td>5.</td><td>a</td><td>$P(X_1 = 2) = \frac{\binom{13}{2}\binom{39}{8}}{\binom{52}{10}} = \frac{\binom{10}{2}\binom{42}{11}}{\binom{52}{13}}$</td><td>b</td><td>$P(X_1 = 2, X_2 = 3, X_3 = 1, X_4 = 7) = \frac{\binom{13}{2}\binom{39}{8}}{\binom{52}{10}} \cdot \frac{\binom{11}{3}\binom{31}{17}}{\binom{42}{20}} \cdot \frac{\binom{8}{1}\binom{14}{5}}{\binom{22}{6}}$</td></tr>
<tr><td>6.</td><td>a</td><td>$P(3A) = \binom{4}{1} \cdot \frac{\binom{13}{3}\binom{13}{0}\binom{13}{0}\binom{13}{0}}{\binom{52}{3}}$</td><td>b</td><td>$P(A, 2B) = (4)_2 \cdot \frac{\binom{13}{1}\binom{13}{2}\binom{13}{0}\binom{13}{0}}{\binom{52}{3}}$</td></tr>
<tr><td></td><td>c</td><td>$P(A, B, C) = \binom{4}{3} \cdot \frac{\binom{13}{1}\binom{13}{1}\binom{13}{1}\binom{13}{0}}{\binom{52}{3}}$</td><td></td><td></td></tr>
</table>

7. a	<table><tr><th>x</th><th>$P(X=x)$</th></tr><tr><td>1</td><td>$10\binom{20}{4}/\binom{200}{4}$</td></tr><tr><td>2</td><td>$\left[90\binom{20}{3}\binom{20}{1}+45\binom{20}{2}^2\right]/\binom{200}{4}$</td></tr><tr><td>3</td><td>$360\binom{20}{2}\binom{20}{1}^2/\binom{200}{4}$</td></tr><tr><td>4</td><td>$210\binom{20}{1}^4/\binom{200}{4}$</td></tr></table>		
b	$P(\text{same number of winning tickets}\mid X=2)=\dfrac{45\binom{20}{2}^2}{90\binom{20}{3}\binom{20}{1}+45\binom{20}{2}^2}$		
8. a	$P(X+Z=0)=\binom{20}{0}\binom{32}{10}/\binom{52}{10}$	b	$P(X\geq 1, Z=0)=\left[\binom{36}{10}-\binom{32}{10}\right]/\binom{52}{10}$
c	$P(X=0, Z\geq 1)=\left[\binom{48}{10}-\binom{32}{10}\right]/\binom{52}{10}$		
d	$P(X+Z=0)+5P(X\geq 1, Z=0)=\binom{32}{10}/\binom{52}{10}+5\left[\binom{36}{10}-\binom{32}{10}\right]/\binom{52}{10}$		
9. a	$P(3\,A, 2\,B)$ $=(13)_2\cdot\binom{4}{3}\binom{4}{2}/\binom{52}{5}$	b	$P(2\,A, 2\,B, 1\,C)$ $=\binom{13}{2}11\cdot\binom{4}{2}\binom{4}{2}\binom{4}{1}/\binom{52}{5}$
10. a	$P(4\,A, 3\,B)$ $=(13)_2\cdot\binom{4}{4}\binom{4}{3}/\binom{52}{7}$	b	$P(4\,A, 2\,B, 1\,C)$ $=(13)_3\cdot\binom{4}{4}\binom{4}{2}\binom{4}{1}/\binom{52}{7}$
c	$P(3\,A, 2\,B, 2\,C)$ $=13\binom{12}{2}\cdot\binom{4}{3}\binom{4}{2}\binom{4}{2}/\binom{52}{7}$	d	$P(2\,A, 2\,B, 1\,C, 1\,D, 1\,E)$ $=\binom{13}{2}\binom{11}{3}\cdot\binom{4}{2}^2\binom{4}{1}^3/\binom{52}{7}$
11. a	$P(X_1=1, X_2=1, X_3=1, X_4=1)=\dfrac{\binom{13}{1}^4}{\binom{52}{4}}$	b	$P(2\,A, 1\,B, 1\,C)=4\binom{3}{2}\cdot\dfrac{\binom{13}{2}\binom{13}{1}^2}{\binom{52}{4}}$
c	$P(2\,A, 2\,B)=\binom{4}{2}\cdot\dfrac{\binom{13}{2}^2}{\binom{52}{4}}$	d	$P(3\,A, 1\,B)=(4)_2\cdot\dfrac{\binom{13}{3}\binom{13}{1}}{\binom{52}{4}}$